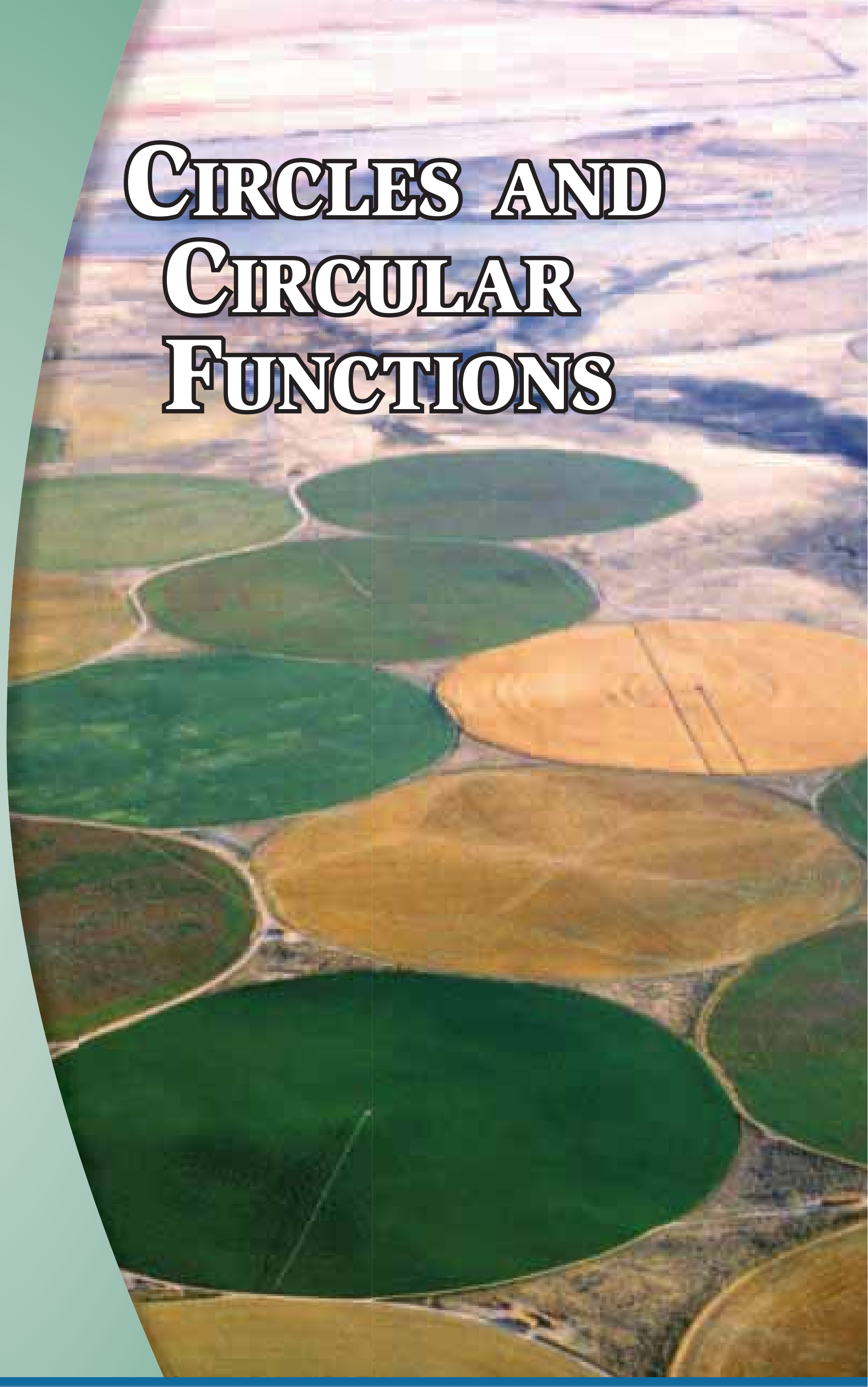
UNIT 6

CIRCLES AND CIRCULAR FUNCTIONS



Circles are the most symmetric of all two-dimensional shapes. As a result, a circle has many interesting properties whose investigation continues to develop visual thinking and reasoning abilities. From another perspective, the rotation of a circle is key to the utility of wheels, circular saws, pulleys and sprockets, center-pivot irrigation, and other mechanisms that involve circles. The motion of the circle as it rotates is a special case of periodic change that can be modeled and analyzed using trigonometric or circular functions.

In this unit, you will learn about and prove properties of special lines and angles as they relate to circles. You will analyze circular motion, both how its power can be harnessed and multiplied in pulley and sprocket systems and how trigonometric functions model the motion. The key ideas will be developed through work on problems in two lessons.

Lessons

1 *Properties of Circles*

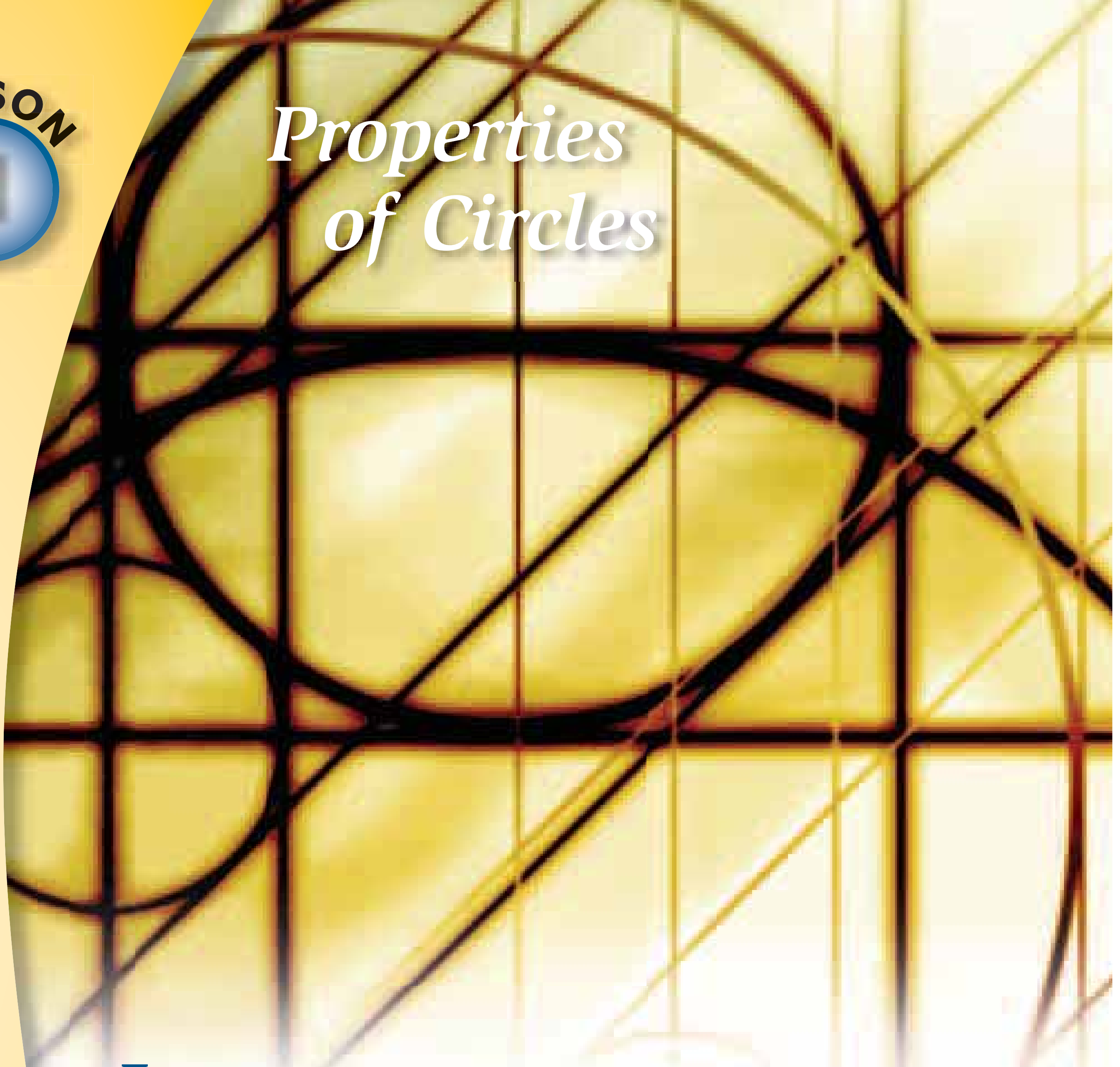
State and prove properties of tangents to circles, chords, arcs, and central and inscribed angles. Interpret and apply these properties.

2 *Circular Motion and Periodic Functions*

Analyze pulley and sprocket systems in terms of their transmission factor, angular velocity, and linear velocity. Use sine and cosine functions of angles and radians to model circular motion and other periodic phenomena.

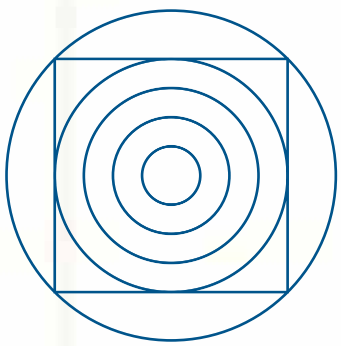
LESSON 1

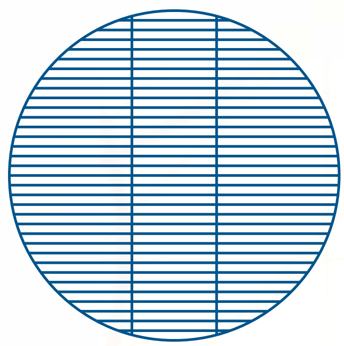
Properties of Circles



In Course 2 of *Core-Plus Mathematics*, you explored some properties of a circle and its center, radius, and diameter. You also learned how to determine the equation of a circle in a coordinate plane.

Circles are the most symmetric of all geometric figures. This symmetry makes them both beautiful and functional. Below, on the left, is a circle-based decorative design from the Congo region of Africa. On the right is a diagram of a circular barbeque grill.





Think About This Situation

Consider the figures on the previous page in light of your study of circles in previous units.

a All points in the plane of a circle with center O either lie on the circle, in its *interior*, or in its *exterior*. For any point P, how can you use the distance OP to define whether P lies on a circle with center O and radius r, in its interior, or in its exterior?

b Suppose point P is in the interior of a circle, and line ℓ contains P. How many points of ℓ, if any, are on the circle? Use a sketch to help explain your answer.

c The five circles in the design on the left are **concentric**, that is, they share the same center. These five circles are obviously similar. Why? Explain why any two circles are similar.

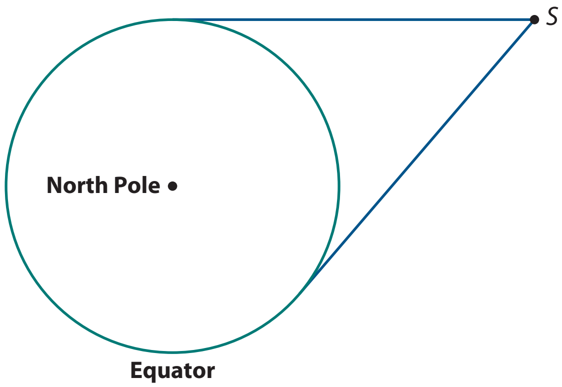
d A **chord** is a line segment that joins any two distinct points on a circle. The barbeque grill shown on the previous page shows several chords. What are the longest chords in a circle? How do you think the length of a chord is related to its perpendicular distance from the center of the circle?

e In the decorative design, the sides of the square are chords of which of the five circles? What can you say about the sides of the square in relation to the second largest circle?

In this lesson, you will explore and verify important properties of special lines and angles in circles. To explore these properties, you may choose to use compass and straightedge, paper folding, measuring, or geometry software. To reason about these properties, you will use congruent triangles, symmetry, coordinate methods, and trigonometric methods. The properties of circles you derive in the first lesson form a basis for the second lesson, which is the study of circular motion and how it is modeled.

Investigation 1 Tangents to a Circle

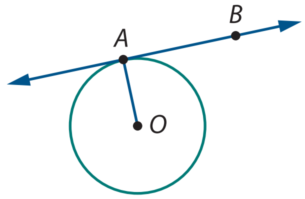
As you learned in Course 2, lines that touch a circle in exactly one point are said to be **tangent** to the circle. Tangent lines have great practical importance. For example, many satellites orbit Earth collecting and transmitting data that is crucial for the operation of modern communication systems. The sketch on the next page shows tangent lines to the circle of the equator from a satellite at point S. These tangent lines can help to determine the range of the satellite's signal.



As you work on problems in this investigation, look for answers to the following question:

What are important properties of tangents to a circle, and how can they be verified?

1 **Tangent at a Point on a Circle** Use a compass, straightedge and protractor, paper folding, or the "Tangents to Circles" sample sketch to investigate properties of a line that is tangent to a circle.



a. Draw a circle like the one at the right with center O and a convenient radius r. Draw a tangent line touching the circle at just one point, A. Point A is called the *point of tangency*. Draw segment $\overline{OA}$. Relative to r, how long is $\overline{OA}$?

b. Suppose B is any point on the tangent line other than A, the point of tangency. Relative to r, how long is $\overline{OB}$? Is B in the interior or exterior of the circle?

c. Find the measure of the angle made by $\overline{OA}$ and the tangent line. Compare your finding with that of others. Write a conjecture about a tangent line and the radius drawn to the point of tangency.

d. Is the converse of your conjecture in Part c also true? If you think so, write your conjecture in if-and-only-if form.

2 As with any if-and-only-if statement, proof of your conjecture in Problem 1 Part d requires an argument in both directions.

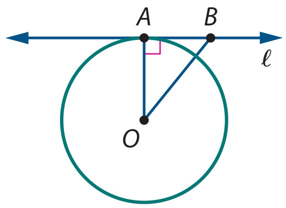
a. One direction is the following:

If a radius is perpendicular to a line at a point where the line intersects the circle, then the line is tangent to the circle at that point.

Students at Peninsula High School wrote the following steps for a proof. Provide reasons that support each of the statements.

Given: Point A on both line ℓ and circle with center O with $\overline{OA} \perp \ell$

Prove: Line ℓ is tangent to the circle with center O, that is, ℓ intersects the circle only in point A.



(1) $\overline{OA}$ is a radius. $\overline{OA} \perp \ell$ at point A.

(2) Let B be any point on line ℓ other than A. Draw $\overline{OB}$.

(3) $\triangle BAO$ is a right triangle with hypotenuse $\overline{OB}$.

(4) $OB > OA$

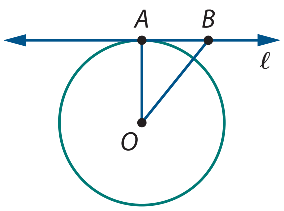
(5) Point B is in the exterior of the circle with center O.

(6) Line ℓ is tangent to the circle with center O.

b. What is the converse of the statement proved in Part a?

c. You are asked to prove the converse in Extensions Task 24. After your conjecture is proved in both directions, it is an if-and-only-if theorem. The theorem should be the same as the statement of the conjecture that you wrote for Problem 1 Part d. If necessary, adjust that statement to make it an accurate statement of the theorem. Add the theorem to your toolkit to use as needed in later problems.

3 Refer to the figure shown at the right.



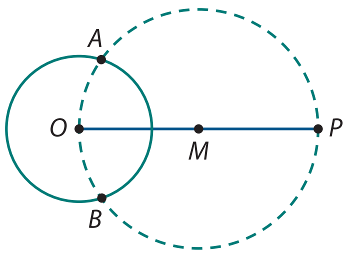
a. If line ℓ is tangent to circle O at point A, the radius of the circle is 4 inches, and $AB = 3$ inches, what is length BO? Explain why.

b. If $AB = 5$ cm, $AO = 12$ cm, and $BO = 13$ cm, why is it correct to conclude that line ℓ must be tangent to the circle at point A?

4 **Tangents from a Point not on a Circle** Given a circle centered at O, here is an algorithm for using a compass and straightedge to construct tangent lines to the circle from a point P in the exterior of the circle.

Step 1. Draw $\overline{OP}$ and then construct its midpoint M.

Step 2. Construct the circle with center M and radius $\overline{OM}$. Label the points of intersection of the two circles A and B.

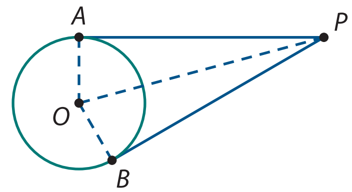


Step 3. Draw lines $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$. These are claimed to be the tangent lines to the circle with center O through point P.

a. On a sheet of paper, draw a large circle with center O and a point P in the exterior of the circle. Use the above algorithm to construct tangent lines to the circle with center O from point P.

b. To verify that this construction works, draw radii $\overline{OA}$ and $\overline{OB}$ and segments $\overline{PA}$ and $\overline{PB}$ on your copy of the figure from Part a. How do you know that $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangent to the circle with center O?

5 Now suppose you are given that $\overline{PA}$ and $\overline{PB}$ are tangent to a circle centered at O. To help prove that $\overline{PA} \cong \overline{PB}$, *auxiliary* line segments $\overline{OA}$, $\overline{OB}$, and $\overline{OP}$ are drawn in the figure.



a. How could you use congruent triangles to prove that $\overline{PA} \cong \overline{PB}$?

b. How could you use the Pythagorean Theorem to show that $\overline{PA} \cong \overline{PB}$?

c. State in words the theorem you have proved about tangents drawn to a circle from an exterior point.

Summarize the Mathematics

In this investigation, you proved and applied some important properties of tangents to a circle.

a Make a list summarizing the properties of tangents that you proved.

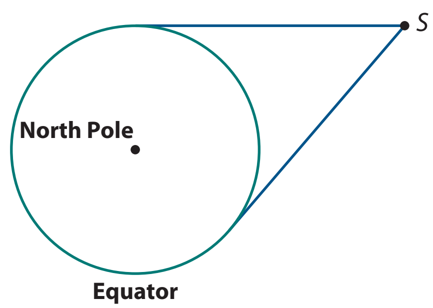
b Some mathematics books define a tangent to a circle with center O at point A to be a line perpendicular to radius $\overline{OA}$ at point A. With that definition, would a tangent line necessarily intersect the circle in exactly one point? Why or why not?

c If point A is any point on a circle with center O, is there always a unique line that is tangent to the circle at point A? Explain.

Be prepared to compare your responses with those of others.

Check Your Understanding

The last three problems of this investigation suggest how to determine measures related to tangents to a circle from an exterior point, such as a satellite orbiting Earth. Suppose a satellite is located in space at point S. In its view of Earth in the plane of the equator, the angle between the lines of sight at S is 50°. The radius of Earth is 3,963 miles.



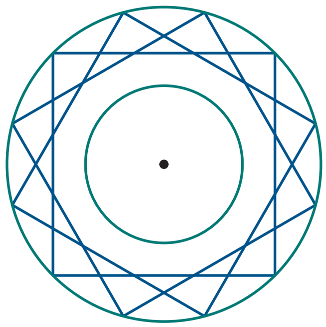
a. What is the distance from S to the horizon along the equator, that is, the length of a tangent from S to Earth's surface along the equator?

b. How high is the satellite S above Earth's surface, that is, the length of a segment S to the closest point on Earth's surface along the equator?

Investigation 2 Chords, Arcs, and Central Angles

As noted at the beginning of this lesson, a **chord** of a circle is a line segment that joins two points of the circle. Any diameter is a chord, but chords may also be shorter than a diameter.

Wheel covers come in a wide range of designs. One wheel cover option is pictured on the left below. Some companies manufacture wheel covers according to buyers' design specifications, as for example, the figure on the right below. Each segment in the design represents a wire that forms a chord that must be at least 3 inches from the center of the wheel cover.

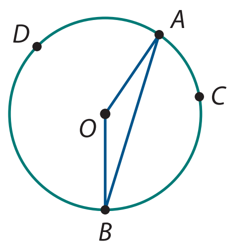


As you work on the problems of this investigation, look for answers to the following questions:

What are important properties of chords, arcs, and central angles of a circle?

How can these properties be proved and applied?

Relating Central Angles, Chords, and Arcs A **central angle** of a circle is an angle of measure less than 180° with vertex at the center of the circle and sides along radii of the circle. In the diagram below, $\angle AOB$ is a central angle and $\overline{AB}$ is the *corresponding chord*.



Each central angle splits a circle into two arcs. In the diagram, the arc $\overset{\frown}{ACB}$ that lies in the interior of the central angle is called a **minor arc**. The other arc $\overset{\frown}{ADB}$ is called a **major arc**. The simple arc notation $\overset{\frown}{AB}$ is used to indicate the minor arc corresponding to the central angle $\angle AOB$.

Arcs are commonly measured in degrees. The degree measure of a minor arc is equal to the measure of the corresponding central angle. The degree measure of a major arc is 360 minus the measure of the corresponding minor arc.

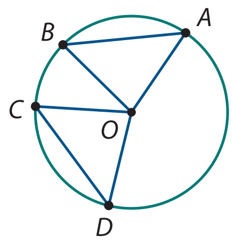
1 Suppose that in the diagram above $m\angle AOB = 160°$.

a. What is $m\overset{\frown}{ACB}$, the measure of the minor arc $\overset{\frown}{ACB}$?

b. What is $m\overset{\frown}{ADB}$, the measure of the major arc $\overset{\frown}{ADB}$?

c. Make a copy of the sketch showing possible locations of a point X so that $m\overset{\frown}{AX} = 90°$?

2 By definition, two arcs have the same measure if and only if their central angles are congruent. But what about the corresponding chords? Consider chords $\overline{AB}$ and $\overline{CD}$ and their central angles pictured at the right.



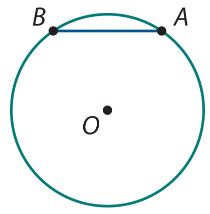
a. Suppose $AB = CD$. Prove that $m\angle AOB = m\angle COD$ and $m\overset{\frown}{AB} = m\overset{\frown}{CD}$.

b. Suppose $m\overset{\frown}{AB} = m\overset{\frown}{CD}$. Prove that $AB = CD$.

c. Summarize in an if-and-only-if statement what you have proved in Parts a and b.

3 **Relationships between Chords** Some other interesting properties of chords can be explored with compass-and-straightedge constructions.

a. Divide the work on these three constructions among your classmates so that each person completes one of the constructions. All three constructions should be done carefully, beginning with a large copy of the figure at the right.



i. Construct the midpoint M of chord $\overline{AB}$. Draw line $\overleftrightarrow{OM}$.

ii. Construct line p, the perpendicular bisector of chord $\overline{AB}$.

iii. Construct line ℓ perpendicular to chord $\overline{AB}$ through point O.

b. Compare lines $\overleftrightarrow{OM}$, p, and ℓ that you and your classmates constructed. How are they alike? How are they different?

c. Based on your findings in the first two constructions, complete the following conjectures.

i. If a line contains the center of a circle and the midpoint of a chord, then ______________.

ii. The perpendicular bisector of a chord of a circle contains ______________.

iii. If a line through the center of a circle is perpendicular to a chord, then ______________.

You are asked to prove these conjectures in the On Your Own tasks.

4 The properties of chords that you explored in this investigation combined with trigonometric methods allow you to find measures of various angles and segments in circles. Suppose that a given circle has radius 6 inches.

a. What is the length of a chord that has a central angle of 115°?

b. What is the measure of the arc of a chord that is 8 inches long? What is the perpendicular distance from the center of the circle to the chord?

c. The perpendicular distance from the center of the circle to a chord is 4 inches. What is the length of the chord? What is the measure of its central angle?

Summarize the Mathematics

In this investigation, you explored and justified some interesting and useful properties of chords in the same or congruent circles.

a Describe how minor and major arcs are determined, named, and measured.

b Summarize how a chord, its minor arc, and its central angle are related.

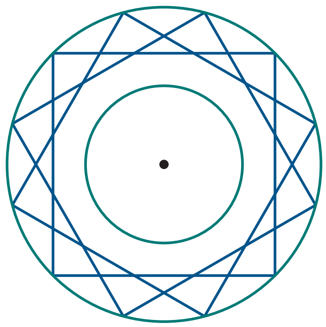
c Complete the following if-and-only-if statement that summarizes two theorems explored in this investigation: A line through the center of a circle bisects a chord if and only if ________.

d If you know the radius of a circle and the length of a chord that is shorter than the diameter, describe how you can determine the measure of the central angle of the chord.

Be prepared to share your responses and thinking with the class.

Check Your Understanding

The designer wheel cover shown at the right has diameter 12 inches. The wires (chords) that are shown are all of equal length.



a. What is the length of each of the wire chords?

b. More wire chords can be added to the design, but none can come closer than 3 inches to the center of the wheel cover.

   i. What is the longest possible wire chord that can be added to this design?

   ii. What is the measure of the central angle of one of these longest wire chords?

Investigation 3 Angles Inscribed in a Circle

Angle measurements and circular arcs are used as navigation aids for ships cruising in dangerous waters. For example, if a stretch of water off two lighthouses holds hazards like rocks, reefs, or shallow areas, the navigation charts might include a circle that passes through the two lighthouses and contains the hazards. It turns out that by carefully tracking the angle determined by the two lighthouses and a ship, its captain can stay safely outside the hazardous area.

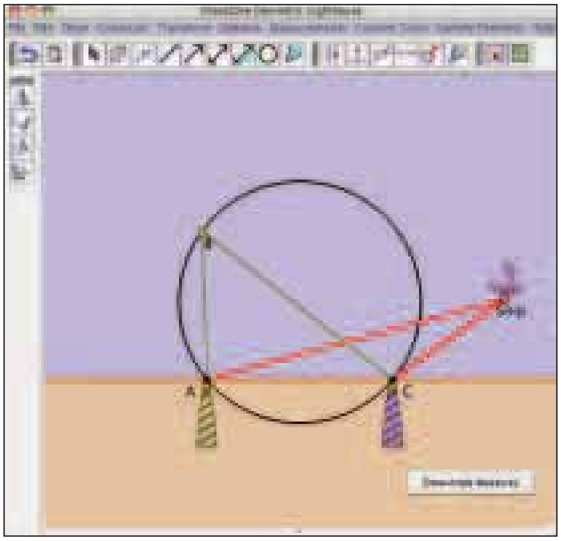
The key to this navigation strategy is the relationship between *inscribed angles* and arcs of circles. As you work on the problems of this investigation, look for answers to the following questions:

What is an inscribed angle in a circle?

How is the measure of an inscribed angle related to the arc it intercepts?

CPMP-Tools

1 Make a drawing like that on the sketch below or use the "Lighthouse" sample sketch in *CPMP-Tools* to see how m∠*ASC* tells whether the ship is inside, outside, or on the circle of hazardous water.



a. First find the approximate measure of $\overset{\frown}{AC}$.

b. What seems to be true of m∠*ASC* when the ship is right on the boundary of the circle of hazardous water?

c. What seems to be true of m∠*ASC* when the ship is inside the circle of hazardous water?

d. What seems to be true of m∠*ASC* when the ship is outside the circle of hazardous water?

e. As captain of the ship, how could you use sighting instruments to keep safely outside the circle of hazardous water?

Inscribed Angles and Intercepted Arcs When the ship is located on the boundary of the region of hazardous water, the angle determined by the two lighthouses and the ship is called an **inscribed angle** in the circle. In general, if points *A*, *B*, and *C* are on the same circle, then those points determine three inscribed angles in that circle.

2 In the figures below, inscribed angle $\angle ABC$ is said to **intercept $\overset{\frown}{ADC}$** where $\overset{\frown}{ADC}$ consists of points A and C and all points on the circle O that lie in the interior of $\angle ABC$. Your goal is to explore the connection between the measures of inscribed angle $\angle ABC$ and its intercepted arc $\overset{\frown}{ADC}$.

Figure 1

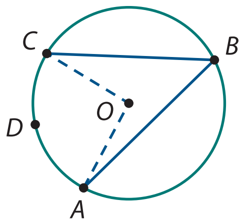


Figure 2

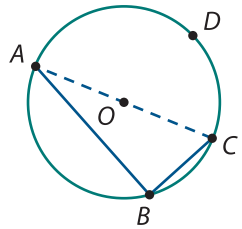
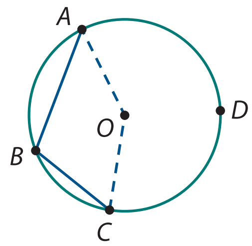


Figure 3



a. When $\overset{\frown}{ADC}$ is a semicircle, as in Figure 2, what is m$\angle ABC$? How do the measures of $\angle ABC$ and $\overset{\frown}{ADC}$ compare?

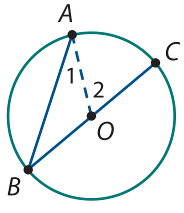
b. Using the "Explore Angles and Arcs" custom tool or by drawing several diagrams, consider multiple instances of Figure 1 where $\angle B$ is an acute angle on the circle. Does the same relationship between the measures of an acute inscribed angle and its intercepted arc that you noted in the case of the inscribed right angle seem to hold?

c. In Figure 3 above, $\angle ABC$ is an obtuse inscribed angle. Its intercepted arc is *major arc* $\overset{\frown}{ADC}$. How can the measure of major arc $\overset{\frown}{ADC}$ be determined if you know m$\angle AOC$? Use the "Explore Angles and Arcs" custom tool to explore the relationship between the measure of obtuse inscribed angle $\angle ABC$ and its intercepted arc $\overset{\frown}{ADC}$.

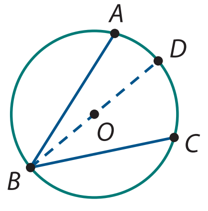
d. Write a conjecture about how the measure of an inscribed angle and the measure of its intercepted arc are related.

3 You noted in Problem 2 Part a that m$\overset{\frown}{ADC} = 180°$ when m$\angle B = 90°$. You also noted in Part c that when the measure of the inscribed angle is more than 180°, you can find the measure of the major arc by subtracting the measure of the minor arc from 360°. This means that to prove the important conjecture, "The measure of an angle inscribed in a circle is half the measure of its intercepted arc," it is sufficient to prove it for acute $\angle B$. To do this, we need to break this case into three smaller cases *all* where $\angle B$ is an *acute* angle as shown in the figures below.

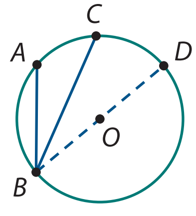
Case 1



Case 2



Case 3



a. These cases differ from one another according to the location of the inscribed angle relative to center O. Explain this difference.

b. A *proof by cases* strategy for proving this conjecture is to prove it for Case 1 first. Then use that result to prove it for the other two cases. Why would carrying out this strategy successfully constitute a complete proof of the conjecture?

4 In Case 1, one side of the inscribed angle contains the center O of the circle. Study the following proof by students at Holland High School of the conjecture in Problem 3. Give a reason why each of the statements is correct.

CPMP-Tools

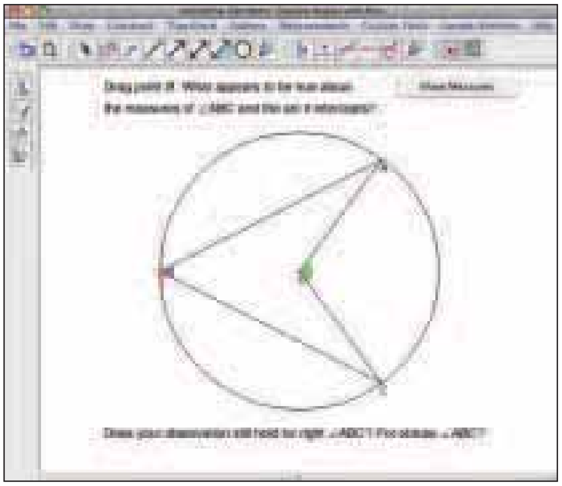
Proof of Case 1:

Center O lies on $\overline{BC}$. Draw radius $\overline{AO}$.	(1)
In $\triangle AOB$, $\overline{AO} \cong \overline{BO}$.	(2)
$m\angle ABC = m\angle 1$	(3)
$m\angle 2 = m\angle ABC + m\angle 1$	(4)
$m\angle 2 = 2(m\angle ABC)$	(5)
$m\angle 2 = m\widehat{AC}$	(6)
$2(m\angle ABC) = m\widehat{AC}$	(7)
$m\angle ABC = \frac{1}{2}(m\widehat{AC})$	(8)

The proof of Case 1 shows that whenever one side of an inscribed angle is a diameter of the circle, its measure is half the measure of its intercepted arc. In Applications Task 8, you will be asked to prove Cases 2 and 3. Assuming all three cases are proved, record the **Inscribed Angle Theorem** in your toolkit.

Angles Intercepting the Same Arc Think back to the introduction of this investigation and the problem of assuring that a ship navigates outside of a circle of hazardous water. Suppose that a captain wants to steer his ship along the boundary of that circle.

5 Use the "Explore Angles and Arcs" custom tool to see how, if at all, the measure of $\angle ABC$ changes as the ship moves.



a. Keeping $m\widehat{AC}$ constant, drag the vertex B on the circle always keeping it on the same side of $\widehat{AC}$. How does the measure of $\angle B$ change as you move it to various locations on the circle?

b. Now change the measure of $\overset{\frown}{AC}$ by dragging point A or point C. Then drag vertex B again. Make a conjecture about the measures of inscribed angles that intercept the same arc.

c. How could the Inscribed Angle Theorem be used to prove this result?

Summarize the Mathematics

In this investigation, you explored the connection between an inscribed angle and its intercepted arc.

a. Summarize the two main properties that you considered in this investigation. How were these properties used to help the captain of the ship avoid the hazardous area?

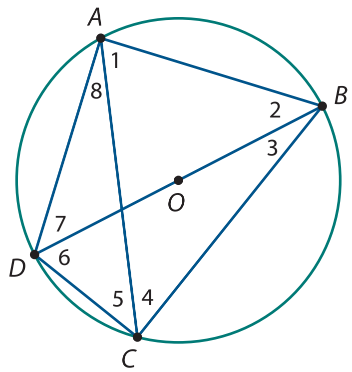
b. If $\angle ABC$ is inscribed in a semicircle, what is the measure of its intercepted arc? What is the measure of $\angle ABC$?

c. A central angle and an inscribed angle intercept the same arc. How are the measures of the angles related?

Be prepared to explain your responses to the class.

Check Your Understanding

In the diagram below, $\overline{BD}$ is a diameter of the circle with center O. Points A, B, C, and D are on the circle.



a. If $m\overset{\frown}{AB} = 100°$, find the measures of as many of the numbered angles as possible.

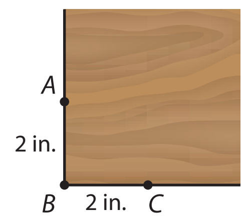
b. Given the two measures $m\angle 1 = 55°$ and $m\angle 2 = 50°$, find the measures of the four minor arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, and $\overset{\frown}{DA}$.

On Your Own

Applications

1 Furniture makers often round corners on tables and cabinets.

a. In the figure at the right, the lower-left corner is to be rounded, using a circle with a radius of 2 inches. Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$ will be tangent to the circle. When the corner is rounded, point *B* will be cut off and points *A* and *C* will lie on the circular arc. Explain how to apply tangent theorems to locate the center of the circle that rounds the corner. Sketch the rounded corner.

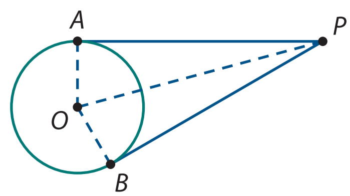


b. Determine the distance from point *B* to the center of the circle.

c. As shown in the figure, point *B* is equidistant from points *A* and *C*. When this corner is rounded as described in Part a, this illustrates a tangent theorem. Which one?

2 Prove the following direct result (*corollary*) of the theorem about tangent segments from a point exterior to a circle.

In the figure at the right, two tangent segments, $\overline{PA}$ and $\overline{PB}$, are drawn to the circle with center *O* from exterior point *P*. What ray appears to be the bisector of $\angle APB$? Prove your answer.



3 Suppose a circle with center *O* has radius 10 inches, *P* is a point in the exterior of the circle, and the tangent segments to the circle from point *P* are $\overline{AP}$ and $\overline{BP}$.

a. If point *P* is 18 inches from the center of the circle, determine the length of $\overline{AP}$ and the measure of $\angle APB$.

b. If $m\angle APB = 48°$, determine the distances *OP*, *AP*, and *BP*.

4 Prove the three conjectures from Investigation 2, Problem 3.

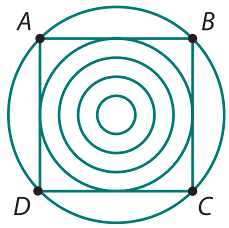
a. *If a line contains the center of a circle and the midpoint of a chord, then the line is perpendicular to the chord.*

b. *The perpendicular bisector of the chord of a circle contains the center of the circle.*

c. *If a line through the center of a circle is perpendicular to a chord, then the line contains the midpoint of the chord.*

5 At the beginning of this lesson, you explored some properties of the decorative design from the Congo region of Africa pictured below. In this design, there are five concentric circles and square *ABCD*. The vertices of *ABCD* lie on the largest circle; that is, the square is inscribed in the largest circle. The sides of the square are tangent to the second largest circle; that is, the square is circumscribed about the second largest circle. In this task, you will continue to explore this design.

a. Suppose the radius of the smallest circle is 0.2 inches. The radii of the next three circles in increasing size are 0.2 inches more than the previous one. What is the length of a side of square *ABCD*?



b. What is the radius of the largest circle?

c. What is $m\overset{\frown}{AB}$? What is $m\overset{\frown}{ABC}$? What is the measure of major arc $\overset{\frown}{ABD}$?

d. Suppose point *E* is the point of tangency of $\overline{AB}$ to the second largest circle. Why must *E* also be the midpoint of $\overline{AB}$?

6 A broken piece of a circular plate shaped like the picture at the right is dug up by an archaeologist, who would like to describe the plate's original appearance and size.

a. Explain how to use one of the theorems that you proved in this lesson to find the location of the center of the plate.

b. Once you find the center, how could you find the circumference of the circular plate?

7 Eighteen equally spaced pins form a "circle" on the geoboard below. The measure of the diameter of circle *O* is 10 cm, and *M* is the midpoint of $\overline{AD}$.

Find the following measures:

a. $m\overset{\frown}{AB}$

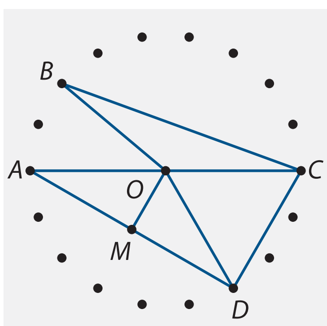
b. $m\overset{\frown}{BC}$

c. *BO*

d. *BC*

e. *CD*

f. $m\angle OMA$



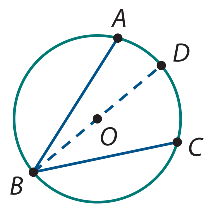
8 Case 2 and Case 3 of the Inscribed Angle Theorem in Investigation 3, Problem 3 are shown below. In each figure, $\overline{BD}$ is a diameter. Assume these two basic properties.

Angle Addition Postulate:
If P is in the interior of $\angle ABC$, then $m\angle ABP + m\angle PBC = m\angle ABC$.

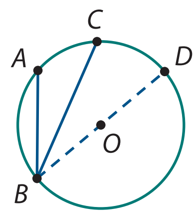
Arc Addition Postulate:
If P is on $\widehat{AC}$, then $m\widehat{AP} + m\widehat{PC} = m\widehat{APC}$.

CPMP-Tools

Case 2



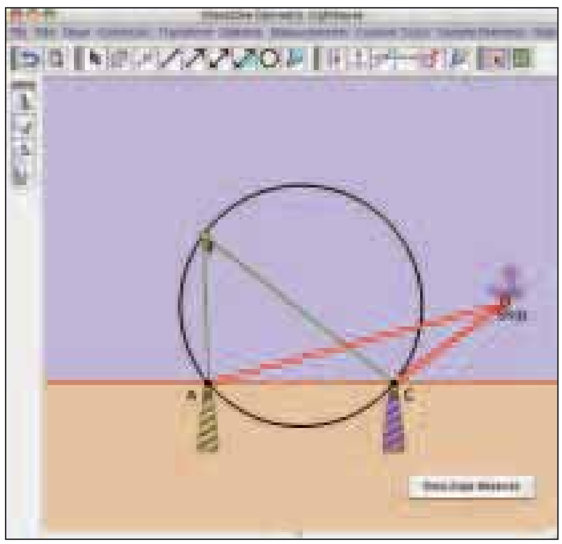
Case 3



a. Prove Case 2: center O is in the interior of inscribed angle $\angle ABC$.

b. Prove Case 3: center O is in the exterior of inscribed angle $\angle ABC$.

9 As you saw in Investigation 3, to protect ships from dangerous rocks as they approach a coastline, the rocks in an area near shore are charted and a circle that contains all of them is drawn. The circle passes through two lighthouses at A and C. The measure of any inscribed angle that intercepts $\widehat{AC}$, such as $\angle ABC$ is called the *horizontal danger angle*.



a. What is the measure of $\widehat{AC}$ in terms of the horizontal danger angle?

b. Suppose the lighthouses at A and C are 0.6 miles apart, and the horizontal danger angle is 35°. What is the radius of the hazardous water circle?

c. What is the perpendicular distance from the center of the hazardous water circle to shore?

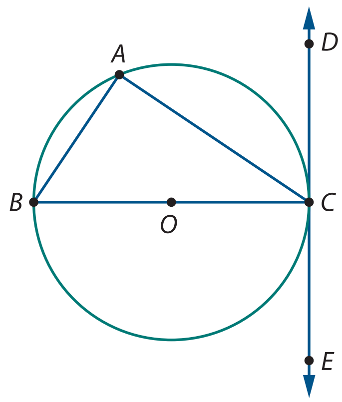
10 A quadrilateral is inscribed in a circle if each vertex is a point on the circle. Prove the theorem:

If a quadrilateral is inscribed in a circle,
then its opposite angles are supplementary.

11 Suppose $\triangle ABC$ is inscribed in a circle with diameter $\overline{BC}$ and $\overleftrightarrow{DE}$ is tangent to the circle at point C.

a. Prove that $m\angle A = m\angle BCE$ and $m\angle B = m\angle DCA$.

b. $\angle DCA$ and $\angle BCE$ are angles formed by a chord of a circle and a tangent line at the endpoint of the chord. $\angle DCA$ intercepts minor arc $\widehat{AC}$, and $\angle BCE$ intercepts semicircle $\widehat{BC}$. Using Part a, what is the relationship between the measures of these angles and of their intercepted arcs? Why?



c. In Part b, you justified the relationship between the measure of an angle formed by a chord and a tangent at an endpoint of the chord and its intercepted arc for acute angle $\angle DCA$ and right $\angle BCE$. Does the same relationship hold true for obtuse angle $\angle ACE$ and its intercepted arc $\widehat{ABC}$? Justify your answer.

d. Parts a, b, and c prove a theorem about the measures of angles formed by a chord of a circle and a tangent line at the endpoint of the chord. Write the theorem and add it to your toolkit.

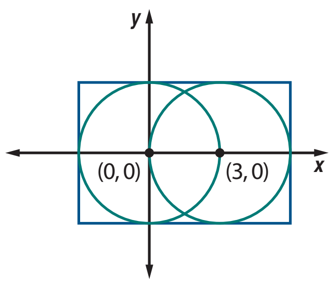
Connections

12 The *interior* and *exterior of a circle* are defined in terms of the circle's radius and center. Consider these ideas from a coordinate perspective.

a. A circle with radius r is centered at the origin. If $P(x, y)$ is any point in the coordinate plane, what conditions must x and y satisfy in each case below?

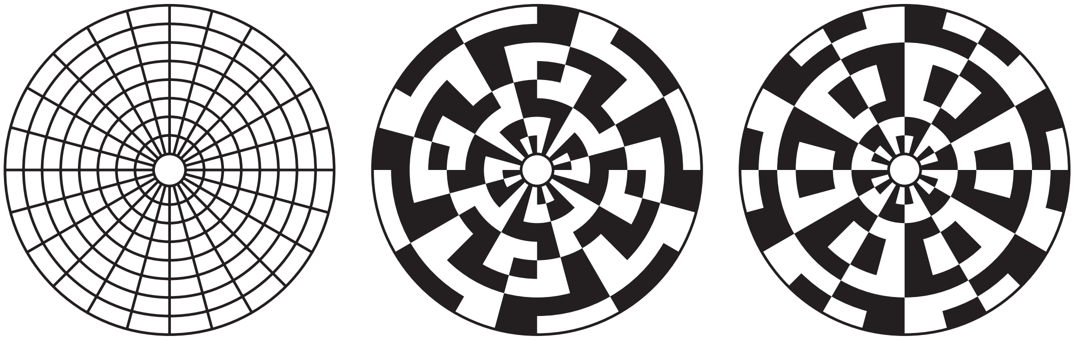
- **i.** P is on the circle.
- **ii.** P is in the interior of the circle.
- **iii.** P is in the exterior of the circle.

b. Two overlapping circles shown at the right are centered at (0, 0) and (3, 0), respectively. They are inscribed in the rectangle as shown. What inequalities must x and y satisfy if $P(x, y)$ is in each of the following locations in the diagram?



- **i.** Inside the rectangle
- **ii.** Inside the left circle but not the right one
- **iii.** Inside both circles
- **iv.** Inside the rectangle but in the exterior of both circles

13 Symmetrical circular black-and-white designs, called Lunda-designs, are used in the Lunda region of Angola. The basis for some of these designs is the left-hand figure. Two of the completed designs are to the right of that figure.



a. In the figure on the left, suppose the radii are extended to the common center of the concentric circles. Twenty-four central angles of equal measures would be formed.

  i. What is the measure of each central angle?

  ii. Choose one central angle that intercepts nine different circles. What are the degree measures of these arcs?

b. In the basis figure on the left, suppose the radius of the innermost concentric circle is 1 cm. As the circles increase in size, their radii are 2 cm, 3 cm, 4 cm, and so on.

  i. What size transformation takes the smallest circle to the largest? The second smallest to the third largest? The second largest to the fifth smallest?

  ii. What is the circumference of the largest circle? What is the length of each arc (of the largest circle) intercepted by the congruent central angles?

c. Describe the symmetries of the middle design above.

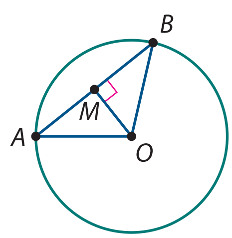
d. Describe the symmetries of the design on the right above.

e. Use an enlarged copy of the base design to create an interesting figure that has rotational symmetry of only 120° and 240°.

14 Two groups of students suggested the following proof plans for this conjecture.

If a line through the center of a circle is perpendicular to a chord, then it bisects the chord.

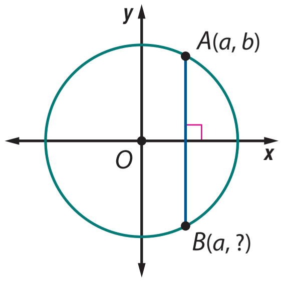
a. Use the following proof plan as a guide to write a proof for this conjecture.



Synthetic Proof Plan—Draw the figure at the right, where $\overline{OM}$ is perpendicular to chord $\overline{AB}$. $\triangle AMO$ and $\triangle BMO$ are congruent right triangles. So, $\overline{MA} \cong \overline{MB}$. Then consider the case when $\overline{AB}$ is a diameter of the circle.

b. The other group decided to write a coordinate proof plan as outlined below. Use their proof plan as a guide to help write an analytic proof for this conjecture.

Analytic Proof Plan—Draw a coordinate axis. Place the center of the circle at the origin and the x-axis perpendicular to chord $\overline{AB}$. Then explain why this placement can be done for any chord. Label point $A(a, b)$ and explain why the x-coordinate of B must be a. Then, reason from the equation of a circle to find the y-coordinate of point B. Finally, show that A and B are the same distance from the x-axis and so the x-axis bisects chord $\overline{AB}$.



c. Which proof, the synthetic or the analytic one, was easier for you to understand? Why?

15 *All-Star Baseball* is a fantasy game that simulates the batting records of current and former all-star baseball players by dividing a circular disk into *sectors* representing hits of various types, walks, strikeouts, fly outs, and ground outs. The ratio of the measure of the central angle of each sector on a particular all-star player's disk to 360° is proportional to his hitting records. For example, a player who hit a double 6% of the times that he batted will have a "double" sector of (0.06)(360°), or 21.6°. When it is his turn to bat in *All-Star Baseball*, a player's disk is fitted on a random spinner. The spinner is spun, and the sector to which the arrow points when it stops determines the outcome of the player's time at bat.

a. Would you expect a player's average hitting performance over a large number of "at-bats" in *All-Star Baseball* to approximate his actual batting record? Why or why not?

b. Babe Ruth was one of the greatest home-run hitters in baseball history. In 1927, he hit 60 home runs, the single season record for many years. He made 691 appearances at bat that year.

i. How many degrees would Ruth's home-run sector for 1927 be in *All-Star Baseball*?

ii. When Babe Ruth (1927 disk) comes to bat in *All-Star Baseball*, what is the probability that he will hit a home run?

c. During the 2007 season, Alex (A-Rod) Rodriguez hit 54 home runs, the most in major league baseball that year. The home run sector on A-Rod's 2007 disk begins at the terminal side of 76.3° and ends at the terminal side of 103.7°.

i. How many times did A-Rod come to bat in 2007?

ii. When Alex Rodriguez (2007 disk) comes to bat in *All-Star Baseball*, what is the probability that he will hit a home run?

16 A circle in a coordinate plane contains points $A(1, 6)$, $B(6, 1)$, and $C(-2, -3)$.

a. Plot these points on graph paper.

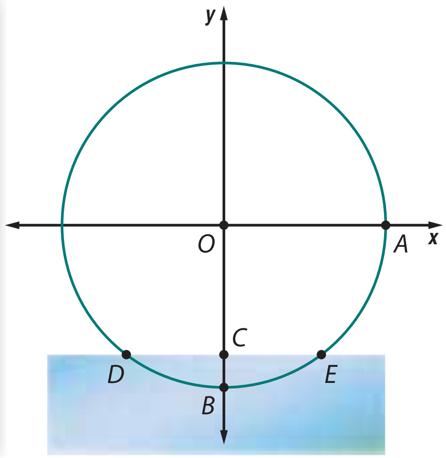
b. Find the coordinates of the center O. What is the radius of the circle?

c. What is the equation of the circle?

d. What is the length of chord $\overline{AC}$?

e. What is the measure of $\angle AOC$? What is the measure of $\angle ABC$?

17 Water wheels are frequently used to generate energy and distribute water for irrigation. The water wheel pictured below operates in a river in the Guangxi Province of China.



In the diagram at the right above, the 30-foot diameter water wheel has been positioned on a coordinate system so that the center of the wheel is the origin and the wheel reaches 3 feet below water level. As the wheel rotates, it takes 12 seconds for point A on the circumference of the wheel to rotate back to its starting position.

a. What are the coordinates of the other labeled points?

b. At its current position, how far is point A from the surface of the water?

c. If the wheel rotates counterclockwise, through what angle does point A rotate to first reach the water at D? How many seconds does it take A to first reach the water?

d. The surface of the water forms a chord $\overline{DE}$. What is the measure of $\overset{\frown}{DE}$? What is the length of chord $\overline{DE}$?

e. What is the measure of inscribed angle $\angle DAE$?

Reflections

18 Describe how to use a compass and straightedge to construct a line tangent to a circle at a point on the circle. This construction is based on a theorem from this lesson. Which one?

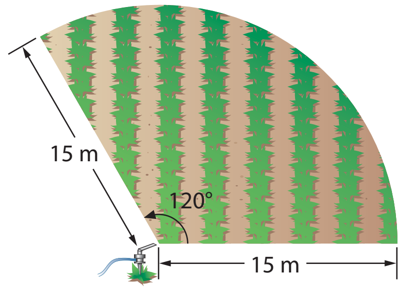
19 Give some examples of tangents to a circle that are evident in bicycle design and use.

20 Summarize the theorems that relate a chord, its midpoint, and the center of a circle. Draw a figure. Explain why these theorems follow from the fact that a circle is symmetric about any line through its center.

21 Use your knowledge about degree measures of arcs and central angles to consider the following questions related to linear measure and area measure. Suppose an irrigation sprinkling system waters an agricultural field as shown in the diagram.

a. What is the area of the watered section of the field which is a sector of the circle with radius 15 meters?

b. If the farmer wishes to fence in this section of field, what length of fencing would be needed?

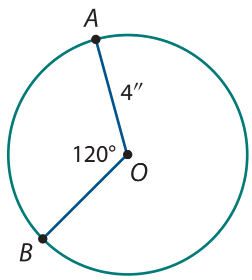


22 Arcs of circles can be measured in degrees as you did in Investigation 2 or in linear measures as when finding the circumference of a circle.

a. In the circle with center O, $AO = 4$", and $m\angle AOB = 120°$, find $m\overset{\frown}{AB}$ in degrees and the length of $\overset{\frown}{AB}$ in inches.

b. Is the following statement true or false? Explain.

Arcs that have the same degree measure also have the same linear measure.



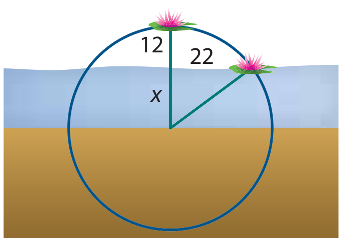
23 How does the arc intercepted by an acute, right, or obtuse central angle of a circle compare to a semicircle of the same circle? Of different circles?

Extensions

24 Prove: *If a line is tangent to a circle, then the line is perpendicular to the radius at the point of tangency.* You may use the following two facts.

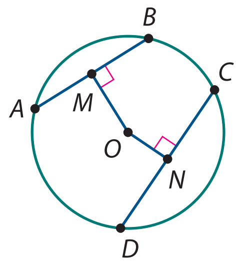
- If a line contains an interior point of a circle, then it intersects the circle in two points.
- The shortest distance from a point to a line is the length of the perpendicular segment.

25 The following is a variation of an ancient Chinese problem. When the stem of a water lily is pulled up so it is vertical, the flower is 12 centimeters above the surface of the lake. When the lily is pulled to one side, keeping the stem straight, the blossom touches the water at a point 22 centimeters from where the vertical stem cuts the surface. The figure at the right is not drawn to scale; the center of the circle has been placed on the bed of the pond. What is the depth x of the water?



26 A familiar pattern in mathematics is to explore, conjecture, and then prove or disprove the conjecture.

a. If two chords of the same circle have different lengths, which is closer to the center of the circle? Explore figures like the one on the right in which one chord, $\overline{CD}$, is longer than the other, $\overline{AB}$. How are their perpendicular distances from the center, ON and OM, related? Write a conjecture.



b. Using right triangles $\triangle AOM$ and $\triangle DON$, prove your conjecture.

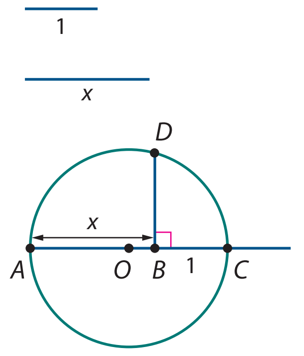
27 Circles and similar triangles provide the foundation for a compass-and-straightedge construction of the square root of the length of any segment. The construction algorithm is given below.

Step 1. Segments of length 1 and x are given. Draw a segment that is greater than $x + 1$. Label one endpoint A.

Step 2. On this segment, mark point B with your compass so that $AB = x$. Then mark point C so that $BC = 1$.

Step 3. Construct the midpoint of $\overline{AC}$. Label the midpoint O. Draw the circle centered at O with radius $OA = OC$.

Step 4. Construct the line perpendicular to $\overline{AC}$ at point B. Label the point D at which the perpendicular line intersects the upper semicircle. Then $BD = \sqrt{x}$, as you will prove in Parts b and c below.

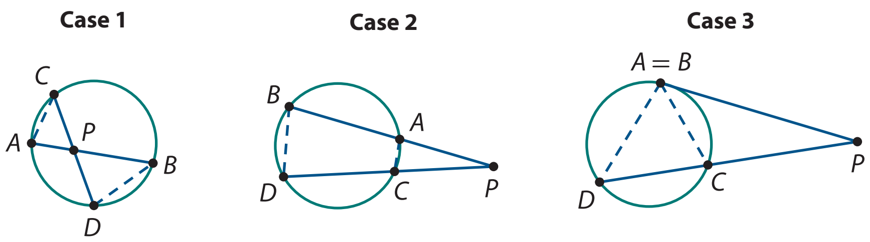


a. To test this construction, draw two segments, one 1 inch long and the other $x = 3$ inches long. Use Steps 1–4 to construct a segment that is $\sqrt{3}$ inches long. Measure the segment with your ruler. What is its length to the nearest eighth of an inch?

b. To justify this construction, draw segments $\overline{AD}$ and $\overline{CD}$. What kind of angle is $\angle ADC$?

c. Show that $\triangle ABD$ is similar to $\triangle DBC$. Then $\frac{x}{BD} = \frac{BD}{1}$. What is the length of $\overline{BD}$ in terms of x?

28 The theorems about angles and chords can help justify interesting conjectures about products of lengths of segments that intersect circles.



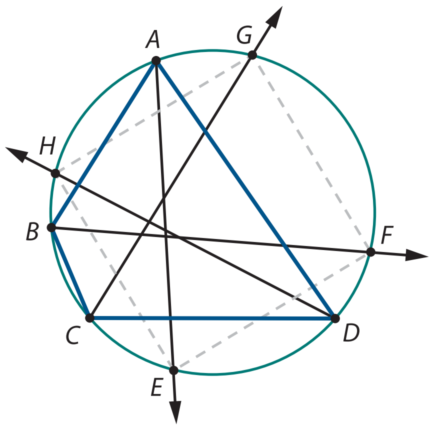
a. The "Chord Lengths in Circles" sample sketch is designed to help you see that for each case above, $PA \times PB = PC \times PD$. Use the sketch to consider multiple instances of each case.

b. In each case shown above, show that $\triangle ACP$ is similar to $\triangle DBP$.

c. Show that the result in Part a implies that $PA \times PB = PC \times PD$. To what does this equation reduce in Case 3?

29 The result at the right is called Keith's Corollary, named for the high school student who discovered it (although it is not a direct result of any theorem). *ABCD* is an arbitrary quadrilateral inscribed in a circle. Each of its four angles is bisected, and the bisectors intersect the circle at the points *E*, *F*, *G*, and *H*.

Prove that *EFGH* is a rectangle. *Hint:* Show that the angles of *EFGH* are right angles by using the facts that opposite angles of quadrilateral *ABCD* are supplementary (see Applications Task 10) and that two inscribed angles intercepting the same arc have equal measure.



Review

30 Write each expression in standard polynomial form.

a. $(3x^3 + 4x^2 - 5x) + (9x^2 - 3x + 12)$

b. $(x^5 - x^3 - 2x - 1) - (5x^5 - x^3 + 5x + 10)$

c. $(x^2 + 3x)(x - 2)$

d. $\dfrac{4x^3 - 2x^2 + 8x}{2x}$

31 Use a compass and straightedge to complete the following constructions.

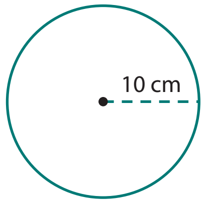
a. Draw a line segment $\overline{AB}$ and construct its perpendicular bisector. Label the midpoint of $\overline{AB}$ as *M*.

b. Draw a line ℓ and mark a point *P* on the line. Construct the line that contains *P* and is perpendicular to line ℓ.

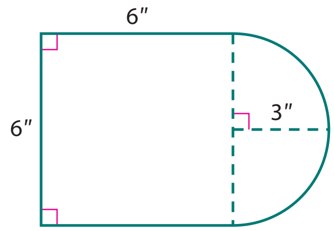
c. Draw a line ℓ and mark a point *P* that is not on the line. Construct the line that contains *P* and is perpendicular to line ℓ.

32 Find the perimeter and area of each figure. Each curved boundary is a circle or a portion of a circle.

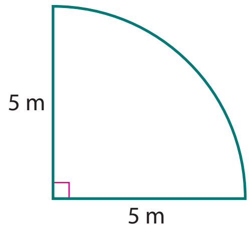
a.



b.



c.



33 Solve each inequality.

a. $-10x + 5 \geq 9 + 7x$

b. $x(x - 5) < 0$

c. $x^2 + 4x > 5$

d. $\frac{-6}{x} < \frac{x + 1}{2}$

34 Quadrilateral $ABCD$ is a rhombus with $AB = 12$ in. and $m\angle BAD = 120°$. Determine each of the following lengths or angle measures.

a. $m\angle ABC$

b. BC

c. $m\angle BAC$

d. AC

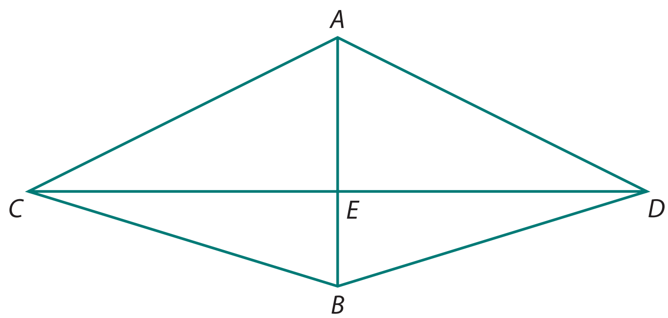
e. BD

35 Rewrite each expression in an equivalent form using only positive exponents.

a. $3x^{-2}(x^4 + 5)$

b. $(4x^3y^{-2})^3$

36 In the diagram below, $\overline{AB}$ is the perpendicular bisector of $\overline{CD}$.



a. Is $\overline{CE} \cong \overline{DE}$? Explain your reasoning.

b. Is $\overline{AC} \cong \overline{AD}$? Explain your reasoning.

c. Is $\triangle CEB \cong \triangle DEB$? Justify your reasoning.

d. What type of quadrilateral is quadrilateral $CBDA$? Justify your response.

37 Solve each equation.

a. $2(3x) = 96$

b. $3x^2 = 96$

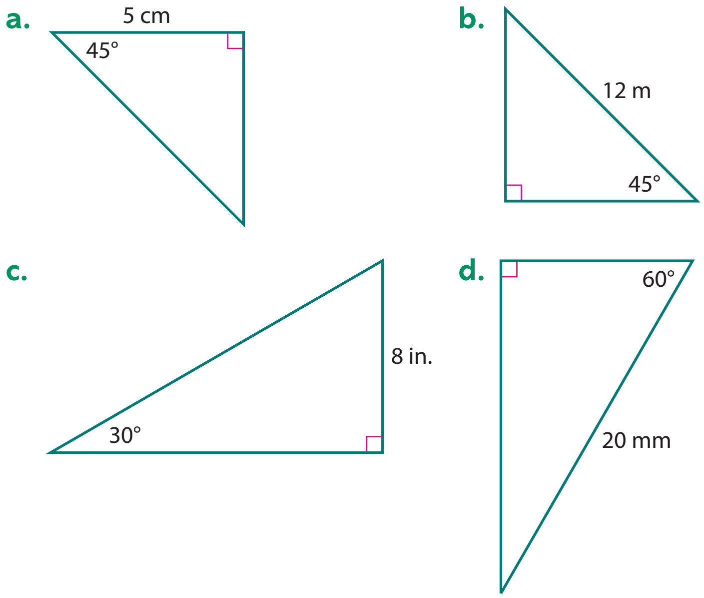
c. $3 + x^2 = 96$

d. $3 + 2x = 96$

e. $3(2x) = 96$

f. $(3 + x)^2 = 96$

38 Find the missing side lengths in each right triangle.



39 If $f(x) = 2x^2 + 11x - 6$, $g(x) = x^2 - 9$, and $h(x) = f(x) \cdot g(x)$, find each of the following.

a. $h(-2)$

b. The degree of $h(x)$

c. All zeroes of $h(x)$

d. The factored form of $h(x)$

e. The y-intercept of $h(x)$

40 Suppose the terminal side of an angle in standard position with measure θ contains the indicated point. Find $\sin \theta$, $\cos \theta$, and $\tan \theta$. Then find the measure of θ to the nearest degree.

a. $P(3, 4)$

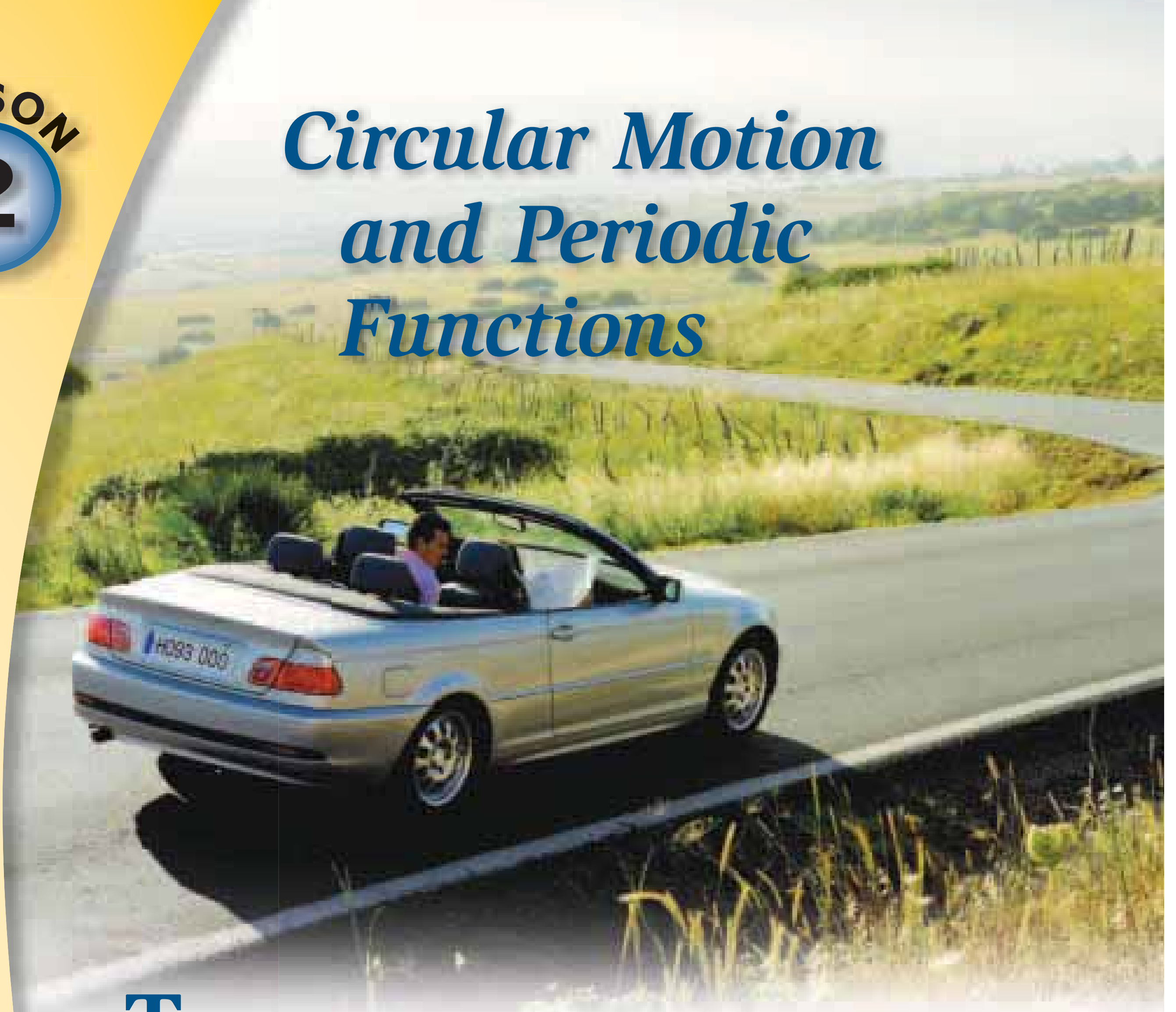
b. $P(3, -4)$

c. $P(-3, 4)$

d. $P(0, -5)$

LESSON 2

Circular Motion and Periodic Functions



The perfect radial symmetry of circles makes them attractive features in decorative designs and works of art. But that defining property is most useful when circular objects like disks, spheres, and cylinders are set in motion. So, it helps to understand mathematical strategies for measuring rates of circular motion and for modeling the (*time, position*) patterns that are generated.

For example, the instrument panel on a high performance car usually contains both a speedometer and a tachometer. The speedometer shows forward speed in miles or kilometers per hour; the tachometer shows revolutions per minute (rpm) by the engine crankshaft.

Think About This Situation

Suppose that the tachometer of a car shows a reading of 3,000 rpm and the tires on that car have diameter of 24 inches, or 2 feet.

a If each revolution of the car's crankshaft were to produce one rotation of the car's wheels, what speed in miles per hour would correspond to the tachometer reading of 3,000 rpm? (Recall that there are 5,280 feet in a mile.)

b The speed implied by your calculations in Part a probably seems far greater than what is reasonable to expect. What do you know about automobile design that explains that fact?

c The tachometer of a typical car will actually show a reading of 3,000 rpm at several different speeds. What do you know about automobile design that explains that fact?

The investigations of this lesson develop strategies for measuring the velocity of circles rotating about axes and rolling along tangent lines. Trigonometric functions are then used to model the relationship between time and position of points on such rotating objects. Finally, those functions are used to describe the *periodic variation* of other phenomena like seasons of the year, pendulums, ocean tides, and alternating electrical current.

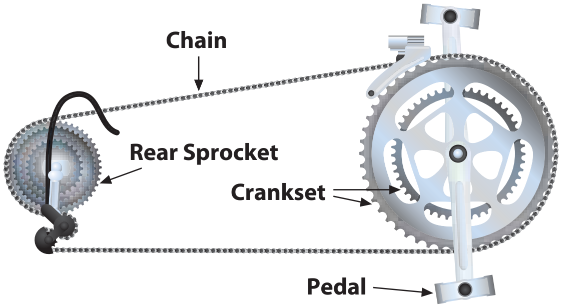
Investigation 1 Angular and Linear Velocity

There are two common measures of circular motion. The angle of rotation about the center of a circle in a unit of time is called its **angular velocity**. It is commonly measured in units like revolutions per minute. The distance traveled by each point on the circle in a unit of time is called its **linear velocity**. It is commonly measured in units like miles per hour or meters per second. As you work on the problems in this investigation, look for answers to the following questions:

How are angular and linear velocity related in circular motion?

How do mechanical systems connect the motion of driver and follower circles?

Bicycle Gear Ratios One familiar example of a transmission connecting moving circles is the gear mechanism used in bicycles. For example, suppose that a mountain bike is set up so that the pedal sprocket in use has 42 teeth and the rear-wheel sprocket in use has 14 teeth of the same size.



1 Consider the given "teeth per sprocket" information above.

a. What does this information tell you about the circumferences of the two sprockets?

b. Suppose that the rider is pedaling at an angular velocity of 80 revolutions per minute.

i. What is the angular velocity of the rear sprocket?

ii. What is the angular velocity of the rear wheel?

2 Suppose that the wheels on the mountain bike have a radius of 33 centimeters.

a. How far does the bike travel for each complete revolution of the 14-tooth rear sprocket?

b. How far does the bike travel for each complete revolution of the front pedal sprocket?

c. If the rider pedals at an angular velocity of 80 rpm, how far will she travel in one minute?

d. How long will the rider need to pedal at 80 rpm to travel 2 kilometers?

3 In the connection between the pedal and rear sprockets of a bicycle, the pedal sprocket is called the **driver** and the rear sprocket is called the **follower**.

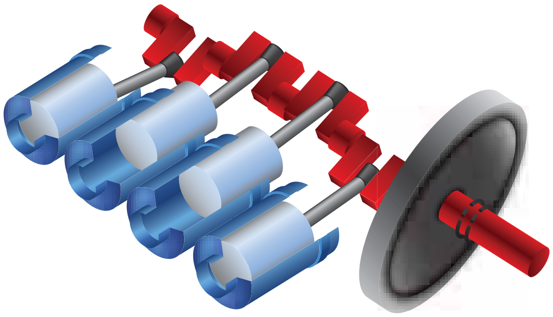
a. Write rules that express the relationships between angular velocities of the driver v_d and follower v_f when the bicycle gears are set in the following ways.

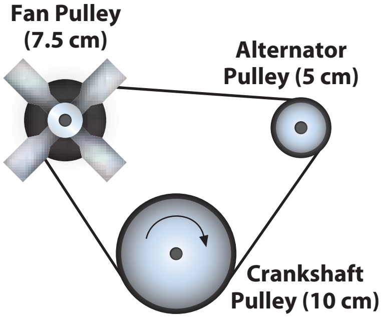
i. Driver sprocket with 30 teeth and follower sprocket with 15 teeth

ii. Driver sprocket with 20 teeth and follower sprocket with 30 teeth

b. Which of the gear settings in Part a will make pedaling harder for the rider? Why?

Crankshafts and Transmissions Many internal combustion engines have pistons that move up and down, causing a cylindrical crankshaft to rotate. The crankshaft *driver* then transfers power to the *follower* wheels but also to other parts of the engine. For example, in many cars, a belt connects the crankshaft to pulleys that drive the fan and the alternator. When the engine is running, the fan cools the radiator while the alternator generates electrical current.





4 The pulley diameters from the diagram above allow us to compare angular velocities of the crankshaft, alternator, and fan pulleys. If the crankshaft makes one complete revolution:

a. how many revolutions will the alternator pulley make?

b. how many revolutions will the fan pulley make?

5 Write rules expressing the relationships of angular velocities for the driver v_d and follower v_f in case the crankshaft is the driver and:

a. the alternator pulley is the follower.

b. the fan pulley is the follower.

6 Suppose that the crankshaft (with diameter 10 cm) of the car is rotating with angular velocity of 1,500 rpm.

a. Calculate the linear velocities in centimeters per minute of points on the edge of each pulley.

i. crankshaft

ii. alternator

iii. fan

b. Explain why the pattern of your answers in Part a makes sense if one thinks about how the belt and pulley mechanism operates.

Summarize the Mathematics

In this investigation, you studied angular and linear velocities—important concepts in the design and analysis of connections between rotating circular objects.

a What do angular and linear velocity measurements tell about the motion of a circular object?

b What are some common units of measurement for angular velocity? For linear velocity?

c How can you use information about the radii of two connected sprockets or pulleys to determine the relationship between their angular and linear velocities?

d If you know the angular velocity and radius of a rotating circular object, how can you determine the linear velocity of points on its circumference?

Be prepared to discuss your responses with the class.

Check Your Understanding

Because power plants that burn fossil fuels are known to increase atmospheric carbon dioxide, many countries are turning to alternative, cleaner sources of electric power. In some places, it makes sense to construct pollution-free wind farms like that pictured below.

a. Suppose that a wind turbine has blades that are 100 feet long and an angular velocity of 2.5 rpm. What is the linear velocity of points at the tip of each blade?

b. Suppose a wind turbine connects a driver pulley with diameter 6 feet to a follower pulley with diameter 4 feet. What is the angular velocity of the follower pulley when the angular velocity of the driver is 3 rpm?

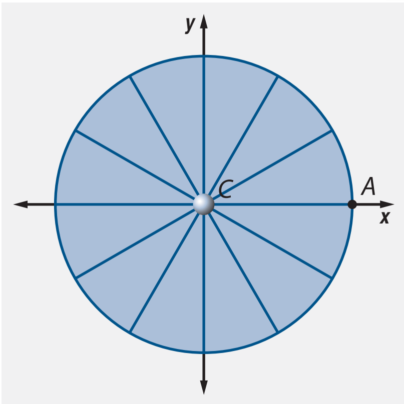
Investigation 2 Modeling Circular Motion

The Ferris wheel was invented in 1893 as an attraction at the World Columbian Exhibition in Chicago, and it remains a popular ride at carnivals and amusement parks around the world. The wheels provide a great context for study of circular motion.

Ferris wheels are circular and rotate about the center. The spokes of the wheel are radii, and the seats are like points on the circle. The wheel has horizontal and vertical lines of symmetry through the center of rotation. This suggests a natural coordinate system for describing the circular motion. As you work on the problems in this investigation, look for an answer to the following question:

How can the coordinates of any point on a rotating circular object be determined from the radius and angle of rotation?

Coordinates of Points on a Rotating Wheel To aid your thinking about positions on a rotating circle, it might be helpful to make a Ferris wheel model that uses a disk to represent the wheel with x- and y-coordinate axes on a fixed backboard.



Connect the disk to the coordinate axis backboard with a fastener that allows the disk to turn freely while the horizontal and vertical axes remain fixed in place.

1. Imagine that a small Ferris wheel has radius 1 decameter (about 33 feet) and that your seat is at point A when the wheel begins to turn counterclockwise about its center at point C.

   a. How does the x-coordinate of your seat change as the wheel turns?

   b. How does the y-coordinate of your seat change as the wheel turns?

2. Find angles of rotation between 0° and 360° that will take the seat from point A to the following special points.

   a. Maximum and minimum distance from the horizontal axis

   b. Maximum and minimum distance from the vertical axis

   c. Points with equal x- and y-coordinates

   d. Points with opposite x- and y-coordinates

When a circle like that modeling the Ferris wheel is placed on a rectangular coordinate grid with center at the origin (0, 0), you can use what you know about geometry and trigonometry to find the x- and y-coordinates of any point on the circle.

3 Find coordinates of points that tell the location of the Ferris wheel seat that begins at point $A(1, 0)$ when the wheel undergoes the following rotations. Record the results on a sketch that shows a circle and the points with their coordinate labels.

a. $\theta = 30°$ **b.** $\theta = 70°$ **c.** $\theta = 90°$

d. $\theta = 120°$ **e.** $\theta = 140°$ **f.** $\theta = 180°$

g. $\theta = 220°$ **h.** $\theta = 270°$ **i.** $\theta = 310°$

4 When the Ferris wheel has rotated through an angle of 40°, the seat that started at $A(1, 0)$ will be at about $A'(0.77, 0.64)$. Explain how the symmetry of the circle allows you to deduce the location of that seat after rotations of 140°, 220°, 320°, and some other angles as well.

5 Suppose that $P(x, y)$ is a point on the Ferris wheel model with $m\angle PCA = \theta$ in degrees.

a. What are the coordinates x and y?

b. How will the coordinate values be different if the radius of the circle is r decameters?

6 With your calculator or computer graphing program set in degree mode, graph the functions $\cos \theta$ and $\sin \theta$ for $0° \leq \theta \leq 360°$. Compare the patterns in those graphs to your ideas in Problems 1 and 2 and to the results of your work on Problem 3.

7 How will the x- and y-coordinates of your seat on the Ferris wheel change during a second complete revolution? How will those patterns be represented in graphs of the coordinate functions for $360° \leq \theta \leq 720°$?

Summarize the Mathematics

In this investigation, you explored the connection between circular motion and the cosine and sine functions.

a Think about the motion of a point on a wheel that starts at (1, 0) and undergoes counterclockwise rotation about the center of the wheel.

i. Describe the pattern of change in the *x*-coordinate of the point as the wheel makes one complete revolution.

ii. Describe the pattern of change in the *y*-coordinate of the point as the wheel makes one complete revolution.

b How are the patterns of change in coordinates extended as the wheel continues to turn counterclockwise through more revolutions?

c Explain how trigonometry can be used to model circular motion.

Be prepared to discuss your responses with the class.

Check Your Understanding

At the Chelsea Community Fair, there is a Ferris wheel with a 15-meter radius. Suppose the Ferris wheel is positioned on an x-y coordinate system as in Problem 1.

a. Danielle is on a seat to the right of the vertical axis halfway to the highest point of the ride. Make a sketch showing the wheel, x- and y-axes through the center of the wheel, and her starting position.

b. Find the x- and y-coordinates of Danielle's seat after the wheel has rotated counterclockwise through an angle of 56°. Through an angle of 130°.

c. Write expressions that give Danielle's position relative to the x- and y-axes for any counterclockwise rotation of θ in degrees from her starting point.

d. Suppose the wheel turns at one revolution every minute. Describe in several different ways the location of Danielle's seat at the end of 50 seconds.

Investigation 3 Revolutions, Degrees, and Radians

The degree is the most familiar unit of measurement for angles and rotations. However, it is common in scientific work to measure angles and rotations in *radians*, a unit that directly connects central angles, arcs, and radii of circles. As you work on the problems of this investigation, look for answers to the following questions:

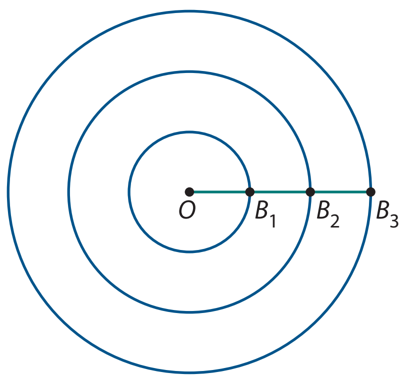
How are angles measured in radians?

How are radians related to other units of measure for angles and rotations?

Radian Measurement Every central angle in a circle intercepts an arc on the circle. The length of that arc is some fraction of the circumference of the circle. This is the key to linking length and angle measurement with radian measure.

1 To see how a *radian* is defined, collaborate with classmates to conduct the following experiment.

a. The diagram at the right represents three concentric circles with center O and radii varying from 1 to 3 inches. Points B_1, B_2, B_3, and O are collinear.



On a full-size copy of the diagram, for each circle:

- wrap a piece of string along the circle counterclockwise from point B_n to a second point A_n so that the length of $\overset{\frown}{A_nB_n}$ is equal to the radius of the circle with radius $\overline{OB_n}$.
- draw $\angle A_nOB_n$ and measure it in degrees.
- record the circle radius and the measure of $\angle A_nOB_n$.

Examine the resulting data and describe any interesting patterns you observe.

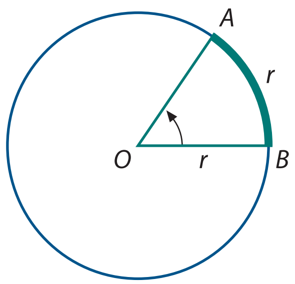
b. What is the approximate degree measure of the central angle you would expect to be determined by a 15-cm arc in a circle with a radius of 15 cm?

c. Suppose a circle with center O has radius r. What would you expect to be the approximate degree measure of $\angle AOB$ if $\widehat{AB}$ is r units long?

CPMP-Tools

A **radian** is the measure of any central angle in a circle that intercepts an arc equal in length to the radius of the circle. To use radian measure effectively in reasoning about angles and rotations, it helps to develop number sense about how radian measures are related to degrees and revolutions.

That is the goal of the next several problems and the "Explore Radians" custom tool in *CPMP-Tools*.



length $\widehat{AB} = r =$ radius
m$\angle AOB = 1$ radian

2 Think about how you measure angles in radians.

a. In a circle with center O, radius 5 centimeters, and $\angle AOB$ intercepting $\widehat{AB}$ of length 10 cm, what are the radian and approximate degree measures of $\angle AOB$?

b. In a circle with center O, radius 8 meters, and $\angle AOB$ intercepting $\widehat{AB}$ of length 24 meters, what are the radian and approximate degree measures of $\angle AOB$?

3 One formula for calculating the circumference of a circle is $C = 2\pi r$. Analyze the ways that Khadijah and Jacy used that fact to find the radian measure equivalent to 120°.

Khadijah

I know $360° = 2\pi$ radians.

Since $\frac{120°}{360°} = \frac{1}{3}$,

the equivalent radian measure

will be $\frac{1}{3}(2\pi)$, or $\frac{2\pi}{3}$.

Jacy

$360° = 2\pi$ radians

$1° = \frac{\pi}{180}$ radians

$120° = 120\left(\frac{\pi}{180}\right)$ radians

$120° = \frac{2\pi}{3}$ radians

a. How do you think Khadijah would find the radian measure equivalent to 30°?

b. How do you think Jacy would find the radian measure equivalent to 30°?

c. Which method, Khadijah's or Jacy's, do you find easiest to use?

d. How do you think Khadijah would reason to find the degree measure equivalent to $\frac{\pi}{4}$ radians?

e. How do you think Jacy would reason to find the degree measure equivalent to $\frac{\pi}{4}$ radians?

4 Use any method you prefer to determine the equivalent angle measures in Parts a–c.

a. Find the measures in radians and revolutions equivalent to these degree measures.

i. 90° **ii.** 150° **iii.** 75° **iv.** 210°

b. Find the measures in degrees and revolutions equivalent to these radian measures.

i. $\frac{\pi}{3}$ **ii.** $\frac{5\pi}{4}$ **iii.** $\frac{\pi}{5}$ **iv.** $\frac{11\pi}{6}$

c. Complete a copy of the following table to show equivalent revolution, degree, and radian measurements. Save the table as a reference for later use.

Revolution/Degree/Radian Equivalents

Revolutions	0	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	1
Degrees	0	30	?	?	90	?	135	150	?	210	?	240	270	300	315	?	360
Radians	?	?	$\frac{\pi}{4}$	$\frac{\pi}{3}$	?	$\frac{2\pi}{3}$	?	?	π	?	$\frac{5\pi}{4}$	?	?	?	?	$\frac{11\pi}{6}$	?

5 Once again, consider the information provided by an engine tachometer.

a. The tachometer of a particular SUV traveling in overdrive on a level road at a speed of 60 mph reads about 2,100 rpm. Find the equivalent angular velocity in degrees per minute and in radians per minute.

b. The idle speed of the SUV is about 1,000 rpm. Find the equivalent angular velocity in degrees per minute and in radians per minute.

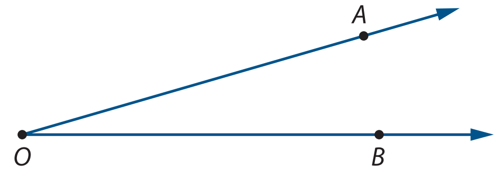
c. Suppose that the SUV engine has an angular velocity of $6{,}000\pi$ radians per minute.

i. What rpm reading would the tachometer show?

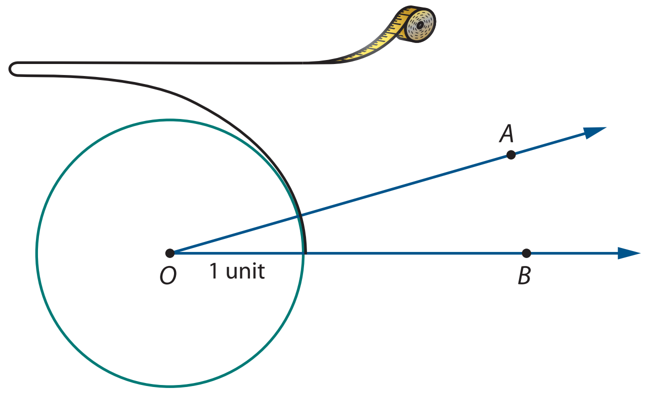
ii. What is the degrees-per-minute equivalent of that angular velocity?

To develop your understanding and skill in use of radian measures for angles and rotations, you might find it helpful to work with the "Explore Radians" custom tool in *CPMP-Tools*. This software shows angles, asks you to estimate the radian and degree measures of those angles, and then checks your estimates numerically and visually.

Linking Angular and Linear Motion The mathematical appeal of radian measure lies in the way that it enables use of linear measurement ideas and tools to produce meaningful measurement of angles and rotations. For example, suppose that you wanted to measure $\angle AOB$ and the only available tool was a tape measure marked off in inches.



To measure the angle in radians, you could draw a circle with center O and radius equal to the unit segment on the tape measure. Then wrap the tape measure around the circle, starting where $\overline{OB}$ meets the circle and noting the point on the tape measure where $\overline{OA}$ meets the circle.



The arc length indicated on the tape measure will be the radian measure of the angle.

The mathematical description of what you have done to measure the angle in radians is to wrap a number line around a circle. If you imagine continuing that wrapping of a number line around a circle (in both directions from the original point of contact), the result is a correspondence between points on the number line and points on the circle.

If you imagine the angle and the circle located on vertical and horizontal coordinate axes, each point has x- and y-coordinates—the cosine and sine of the measure of the angle, respectively. In this way, it is possible to define these trigonometric functions with domain all real numbers.

6 Consider first the cosine function, the correspondence between points on the number line and the x-coordinates of points on the unit circle.

a. What pattern of change would you expect in values of the cosine function as the positive number line is wrapped in a counterclockwise direction around the unit circle starting at (1, 0)? How would that pattern appear in a plot of $(t, \cos t)$ values?

b. What pattern of change would you expect in values of the cosine function as the negative real number line is wrapped in a clockwise direction around the unit circle starting at (1, 0)? How would that pattern appear in a plot of $(t, \cos t)$ values?

c. Set your graphing calculator or computer software in radian mode. Graph cos t for $-2\pi \leq t \leq 2\pi$ to test your ideas in Parts a and b.

d. What would you expect to appear in the graphing window if the lower and upper bounds for t were reset to $-4\pi \leq t \leq 4\pi$?

e. Why were the suggested upper and lower bounds for the graphing windows multiples of 2π?

7 Consider next the sine function, the correspondence between points on the number line and the y-coordinates of points on the unit circle.

a. What pattern of change would you expect in values of the sine function as the positive number line is wrapped in a counterclockwise direction around the unit circle starting at (1, 0)? How would that pattern appear in a plot of (t, *sin* t) values?

b. What pattern of change would you expect in values of the sine function as the negative real number line is wrapped in a clockwise direction around the unit circle starting at (1, 0)? How would that pattern appear in a plot of (t, *sin* t) values?

c. Set your graphing calculator or computer software in radian mode. Graph sin t for $-2\pi \leq t \leq 2\pi$ to test your ideas in Parts a and b.

d. What would you expect to appear in the graphing window if the lower and upper bounds for t were reset to $-4\pi \leq t \leq 4\pi$?

e. In Parts c and d, the upper and lower bounds given for the graphing windows are multiples of 2π. What are advantages of this form?

Summarize the Mathematics

In this investigation, you learned about measuring angles using radians. The radian is a unit for measuring angles that is useful because it connects length and angular measurement.

a Describe how to draw an angle that measures 1 radian.

b How are angle measurements in revolutions, radians, and degrees related to each other?

c If an angle has measure r in radians, d in degrees, and v in revolutions what rule expresses:

- **i.** r as a function of d?
- **ii.** r as a function of v?
- **iii.** d as a function of r?
- **iv.** v as a function of r?

d How are the cosine and sine functions defined for radian measures or real numbers? How are the resulting numeric and graphic patterns of values for those functions similar to or different from those when the functions are defined for degrees?

Be prepared to discuss your responses with the class.

Check Your Understanding

Suppose that one of the wind turbines shown in the Check Your Understanding of Investigation 1 has a driver pulley with 6-foot diameter attached by a belt to a follower pulley with 4-foot diameter.

a. If the driver pulley turns at 3 revolutions per minute:

i. what is the angular velocity of the driver pulley in radians per minute?

ii. what is the angular velocity of the follower pulley in radians per minute?

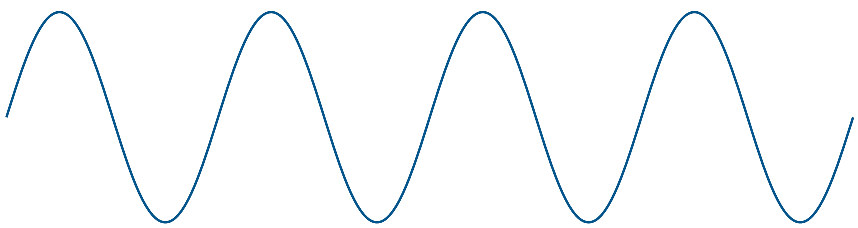
c. Suppose that economic use of the wind turbine to generate electric power requires the follower pulley to turn at a rate of 90π radians per minute. How fast, in radians per minute, must the driver pulley turn to accomplish this rate?

d. Draw sketches and use geometric reasoning to find exact values of the sine and cosine functions when the input values are the following radian measures or real numbers. Check your answers using technology or the table you completed in Problem 4.

i. $\frac{3\pi}{4}$ **ii.** π **iii.** $\frac{7\pi}{6}$ **iv.** $\frac{5\pi}{3}$

Investigation 4 Patterns of Periodic Change

Tracking the location of seats on a spinning Ferris wheel shows how the cosine and sine functions can be used to describe rotation of circular objects. Wrapping a number line around a circle shows how the radius of the circle can be used to measure angles, arcs, and rotations in radians and how the cosine and sine can be defined as functions of real numbers. In both cases, the resulting functions had repeating graph patterns that are called *sinusoids*.



Many important variables that change over time have sinusoidal graphs. As you work on the problems of this investigation, look for answers to the following question:

How can the cosine and sine functions be modified to represent the variety of important patterns of periodic change?

A Family of Ferris Wheels The Ferris wheel that you analyzed during work on Investigation 2 had a radius of one decameter and an unspecified angular velocity. To model the motion of wheels with different radii and to account for different rates of rotation, it is necessary to construct functions that are variations on the basic cosine and sine functions.

1 If a Ferris wheel has radius 1.5 decameters (15 meters), the functions 1.5 cos θ and 1.5 sin θ give *x*- and *y*-coordinates after rotation of θ for a seat that starts at (1.5, 0). Compare the graphs of these new coordinate functions with the graphs of the basic cosine and sine functions.

a. Find the maximum and minimum points of the graphs of cos θ and 1.5 cos θ when:

i. θ is measured in degrees.

ii. θ is measured in radians.

b. Find the θ-axis intercepts of the graphs of cos θ and 1.5 cos θ when:

i. θ is measured in degrees.

ii. θ is measured in radians.

c. Find the maximum and minimum points of the graphs of sin θ and 1.5 sin θ when:

i. θ is measured in degrees.

ii. θ is measured in radians.

d. Find the θ-axis intercepts of the graphs of sin θ and 1.5 sin θ when:

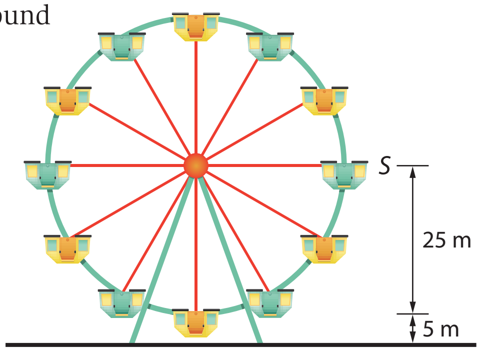
i. θ is measured in degrees.

ii. θ is measured in radians.

e. How would the maximum and minimum points and the θ-axis intercept change if the Ferris wheel being modeled had radius *a* and the coordinate functions were *a* cos θ and *a* sin θ?

2 When riding a Ferris wheel, customers are probably more nervous about their height above the ground than their distance from the vertical axis of the wheel.

Suppose that a large wheel has radius 25 meters, the center of the wheel is located 30 meters above the ground, and the wheel starts in motion when seat *S* is at the "3 o'clock" position.



a. Modify the sine function to get a rule for $h(\theta)$ that gives the *height* of seat *S* in meters after rotation of θ. Compare the graph of this height function with the graph of sin θ.

b. Find the maximum and minimum points on the graphs of sin θ and $h(\theta)$ when:

i. θ is measured in degrees.

ii. θ is measured in radians.

c. Find the θ-axis intercepts on the graphs of $\sin\theta$ and $h(\theta)$.

d. How would the maximum and minimum points and the θ-axis intercept change if the Ferris wheel being modeled had radius a and its center was c meters above the ground? Why is $c > a$?

Functions with sinusoidal graphs all have patterns that repeat as values of the independent variable increase. Functions that have that kind of repeating pattern are called **periodic functions**. The **period** is the length of the shortest interval of values for the independent variable on which the repeating pattern of function values occurs. If a periodic function has maximum and minimum values, half the absolute value of the difference between those values is called its **amplitude**. In addition to the change in amplitude of $\sin\theta$ used to model Ferris wheel motion, the graph of $h(\theta)$ appears to involve a shift of that graph upward. That upward shift of 30 meters is called the ***y*-displacement**.

3 The functions $\cos\theta$, $\sin\theta$, $1.5\cos\theta$, $1.5\sin\theta$, and $h(\theta)$ all have the same period. Find that common period and the amplitude of each function when:

a. θ is measured in degrees.

b. θ is measured in radians.

4 How are the period and the amplitude of $f(\theta) = a\cos\theta + c$ and $g(\theta) = a\sin\theta + c$ related to the values of the numbers a and c?

Period and Frequency In various problems of this lesson, you have seen how to find the location of points on a rotating circle from information about the starting point and the angle of rotation. For actual rotating wheels, it is much easier to measure elapsed *time* than the *angle* of rotation. The link between these two variables is *angular velocity*.

For constant angular velocity, the angle of rotation varies directly with the elapsed time. So, it is often convenient to describe the location of points on a wheel as a function of time. Because radian measure connects points on a number or time line axis to points on a circle with a common unit of measurement, it is customary to use radian measure in analyzing circular motion as a function of time.

5 Suppose that the height of a Ferris wheel seat changes in a pattern that can be modeled by the function $h(t) = 25\sin t + 30$, where time is in minutes and height is in meters.

a. What are the period and amplitude of $h(t)$? What do those values tell about motion of the Ferris wheel?

b. If a seat starts out in the "3 o'clock" position, how long will it take the seat to first return to that position? At what times will it revisit that position?

c. What is the angular velocity of this wheel:

i. in revolutions per minute? (This is also called the **frequency** of the periodic motion.)

ii. in degrees per minute?

iii. in radians per minute?

6 Suppose that the height (in meters) of seats on different Ferris wheels changes over time (in minutes) according to the functions given below. For each function:

- find the height of the seat when motion of the wheel begins.
- find the amplitude of $h(t)$. Explain what it tells about motion of the wheel.
- find the period of $h(t)$. Explain what it tells about motion of the wheel.
- find the angular velocity of the wheel in revolutions per minute—the frequency of $h(t)$.

CPMP-Tools

a. $h(t) = 15 \sin 0.5t + 17$ **b.** $h(t) = 24 \cos 2t + 27$

c. $h(t) = 12 \sin 1.5t + 13$ **d.** $h(t) = -12 \cos t + 14$

7 In Problems 5 and 6, you analyzed properties of functions with rules in the general form

$$h(t) = a \sin bt + c \quad \text{and} \quad h(t) = a \cos bt + c.$$

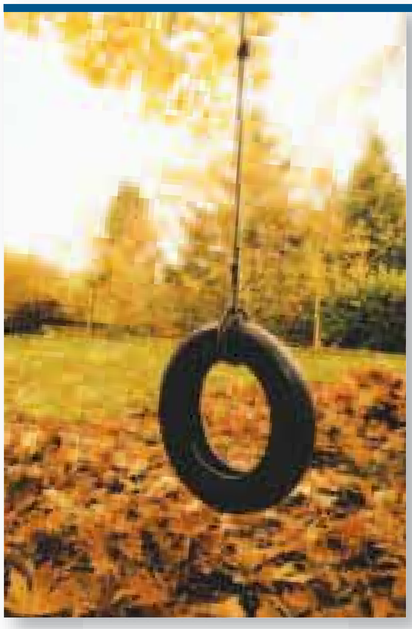
a. How do each of the values a, b, and c seem to affect the graphs of the functions?

b. Check your ideas by reasoning about the function rules and by graphing examples with a calculator or by using the parameter variation capability of *CPMP-Tools* CAS. Be sure your answer accounts for both positive and negative values of a and c.

Modeling Periodic Change over Time Variations of the basic cosine and sine functions can be used to model the patterns of change in many different scientific and practical situations. The next three problems ask you to analyze familiar situations.

8 Pendulums are among the simplest but most useful examples of periodic motion. Once set in motion, the arm of the pendulum swings left and right of a vertical axis. The angle of displacement from vertical is a periodic function of time that depends on the length of the pendulum and its initial release point.

Suppose that the function $d(t) = 35 \cos 2t$ gives the displacement from vertical (in degrees) of the tire swing pendulum shown at the right as a function of time (in seconds).



a. What are the amplitude, period, and frequency of $d(t)$? What does each tell about the motion of the swing?

b. If the motion of a different swing is modeled by $f(t) = 45 \cos \pi t$, what are the amplitude, period, and frequency of $f(t)$? What does each tell about the motion of that swing?

c. Why does it make sense to use variations of the circular function $\cos t$ to model pendulum motion?

d. What function $g(t)$ would model the motion of a pendulum that is released from a displacement of 18° right of vertical and swings with a frequency of 0.25 cycles per second (a period of 4 seconds)?

9. At every location on Earth, the number of hours of daylight varies with the seasons in a predictable way. One convenient way to model that pattern of change is to measure time in days, beginning with the spring equinox (about March 21) as $t = 0$. With that frame of reference, the number of daylight hours in Boston, Massachusetts is given by $d(t) = 3.5 \sin \frac{2\pi}{365}t + 12.5$.

a. What are the amplitude, period, and frequency of $d(t)$? What do those values tell about the pattern of change in daylight during a year in Boston?

b. What are the maximum and the minimum numbers of hours of daylight in Boston? At what times in the year do they occur?

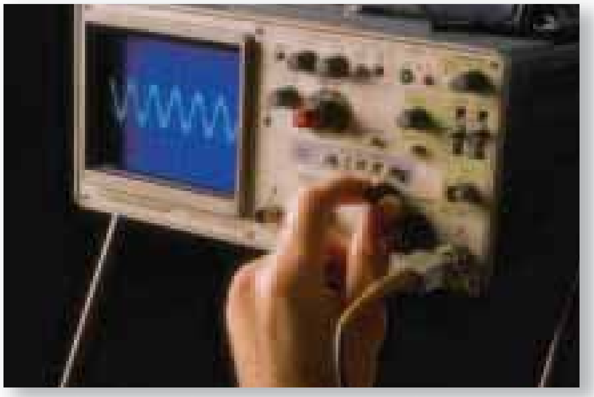
c. If the function giving the number of daylight hours in Point Barrow, Alaska, had the form $f(t) = a \sin bt + c$, how would you expect the values of a, b, and c to be related to the corresponding numbers in the rule giving daylight hours in Boston?

d. Why does it make sense that the function giving daylight hours at points on Earth should involve the circular function $\sin t$?

10. Oscilloscopes are common scientific instruments for display of the periodic variation in pressure and electronic waves that carry sound and light. For example, the function

$$V(t) = 170 \sin 120\pi t$$

gives the voltage in a standard alternating current electrical line as a function of time in seconds.



a. What maximum and minimum voltage are implied by the function $V(t)$?

b. What period and frequency are implied by the rule for $V(t)$? (That standard frequency is called **one Hertz**.)

Summarize the Mathematics

In this investigation, you explored the use of the sine and cosine functions to represent periodic phenomena.

a. What is the defining property of a periodic function?

b. What are the amplitude and frequency of a periodic function?

c. What are the period, amplitude, and frequency of $f(t) = \sin t$ and $g(t) = \cos t$?

d. How can the values of a, b, and c be used to find the amplitude, period, and frequency of $f(t) = a \sin bt + c$ and $g(t) = a \cos bt + c$?

e. What are some common examples of periodic change over time?

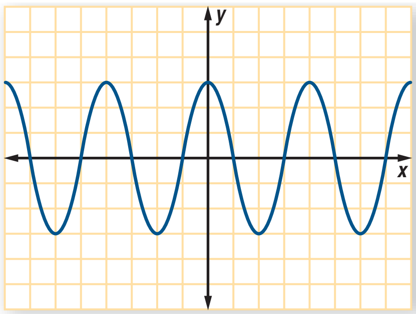
Be prepared to discuss your responses with the class.

Check Your Understanding

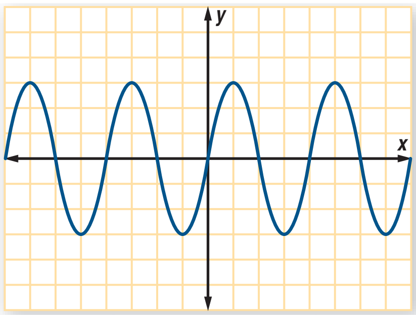
Portions of periodic graphs are shown below with windows $-4\pi \leq x \leq 4\pi$ and $-6 \leq y \leq 6$. Without using technology, match each graph to one of the given functions. (Not all function rules will be used.) In each case, explain the reason for your choice.

I	$y = 3 \sin x$	**II**	$y = 3 \cos x$	**III**	$y = 3 + \sin x$
IV	$y = -3 \sin x$	**V**	$y = -3 \cos x$	**VI**	$y = \sin 3x$

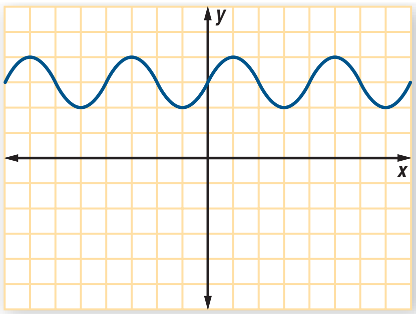
a.



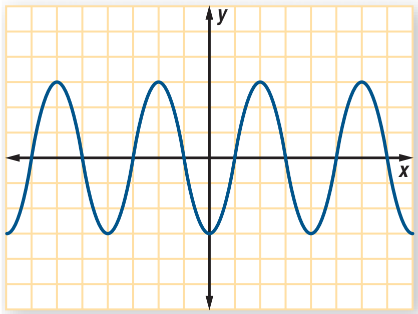
b.



c.



d.



On Your Own

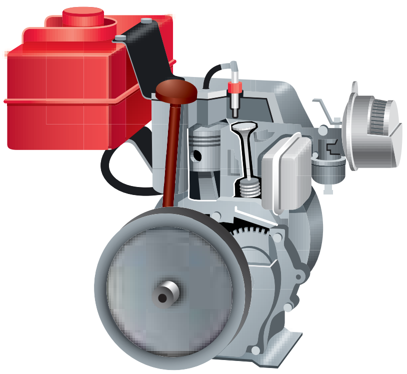
Applications

1 Lena's mountain bike has 21 speeds. To get started, she shifts gears so that the chain connects the 42-tooth crankset with a 28-tooth rear-wheel sprocket.

a. If Lena pedals at 40 rpm, at what rate do the rear sprocket and wheel turn?

b. If Lena pedals at 40 rpm, how far will she travel in one minute if her bike has tires with 66-cm diameters?

c. What is the relationship between the angular velocities of the pedal sprocket v_p and the rear wheel sprocket v_w?



2 In go-carts, the engine driver sprocket is attached to a follower sprocket on the rear axle by a belt. These sprockets can have many different diameters depending upon course demands and safety.

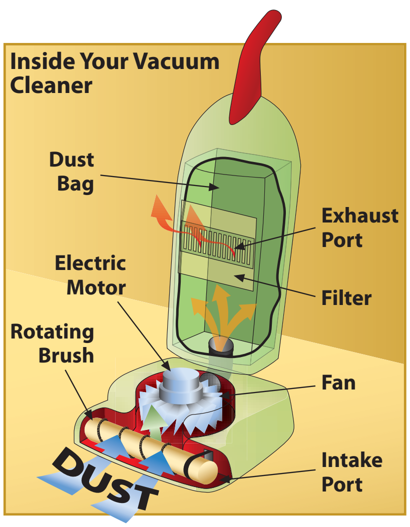
a. Sketch the situation in which an engine sprocket with a 7-cm diameter drives a rear-axle sprocket with a 10-cm diameter.

b. Suppose that the engine is turning at 1,620 rpm.

i. Find the angular velocity of the rear axle.

ii. Find the go-cart's speed, in cm per minute and km per hour, if the rear wheels have a diameter of 28 cm.

c. When rounding corners, a speed of 30 km per hour or less is needed to reduce lateral sliding. What engine speed in rpm is optimal?



3 The operation of most vacuum cleaners depends on circular motion at several points in the vacuuming process. In one vacuum cleaner model, the rotating brush is 1.5 inches in diameter and is driven by a rubber belt attached to a driver pulley on the motor. The driver pulley is 0.5 inches in diameter. The belt is attached with a half twist, so that the rotating brush and the driver rotate in opposite directions.

a. The rotating brush is designed to help pull dust from the floor into the vacuum cleaner. Should the rotating brush be turning clockwise or counterclockwise? Explain.

b. Sketch this pulley system.

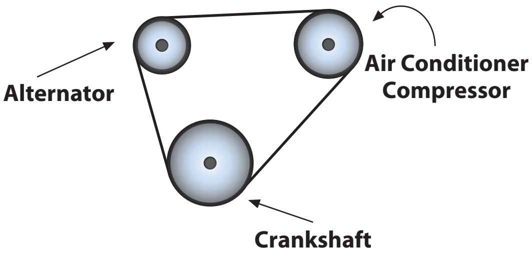
c. What are the circumferences of the driver pulley and the rotating brush?

d. How far does a point on the edge of the rotating brush travel in one revolution of the driver?

e. If the driver pulley is turning 600 rpm, how fast is the rotating brush turning?

f. Write a rule that shows the relationship between angular velocities of the engine drive shaft v_D and the rotating brush v_B.

4 The crankshaft of a particular automobile engine has an angular velocity of 1,500 rpm at 30 mph. The crankshaft pulley has a diameter of 10 cm, and it is attached to an air conditioner compressor pulley with a 7-cm diameter and an alternator pulley with a 5-cm diameter.

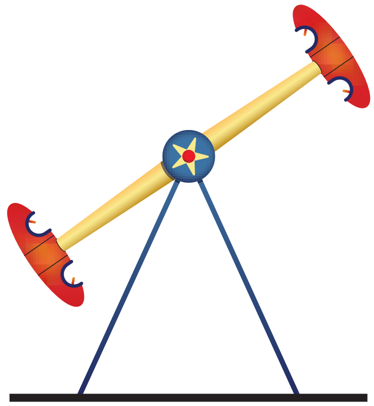


a. At what angular velocities do the compressor and alternator turn?

b. The three pulleys are connected by a 60-cm belt. At a crankshaft rate of 1,500 rpm, how many times will the belt revolve through its 60-cm length in one minute?

c. Most belts do not show significant wear until each point of the belt has traveled about 20,000 kilometers. How long can the engine run at 1,500 rpm before the belt typically would show wear?

5 One of the scarier rides at carnivals and amusement parks is provided by a long rotating arm with riders in capsules on either end. Of course, the capsules not only move up and down as the arm rotates, but they spin other ways as well.



Suppose that the arm of one such ride is 150 feet long and that you get strapped into one of the capsules when it is at ground level. Assume simple rotating motion and treat the capsule as a single point. Find your height above the ground when the arm has made the following rotations counterclockwise from its starting vertical position.

a. 20° **b.** 45°

c. 85° **d.** 90°

e. 120° **f.** 180°

g. 270° **h.** 300°

i. 340°

6 Radio direction and ranging (radar) is one of the most widely used electronic sensing tools. Most of us probably know about radar from its applications in measuring speed of baseball pitches and automobiles. But it is also an invaluable tool in navigation and weather forecasting.

In those applications, the echoes to a rotating transmitter/receiver are displayed as blips on a scope. Each blip is located by distance and angle.

To describe locations of radar blips by distance east/west and north/south of the radar device, the (*distance, angle*) information needs to be converted to rectangular (x, y) coordinates.

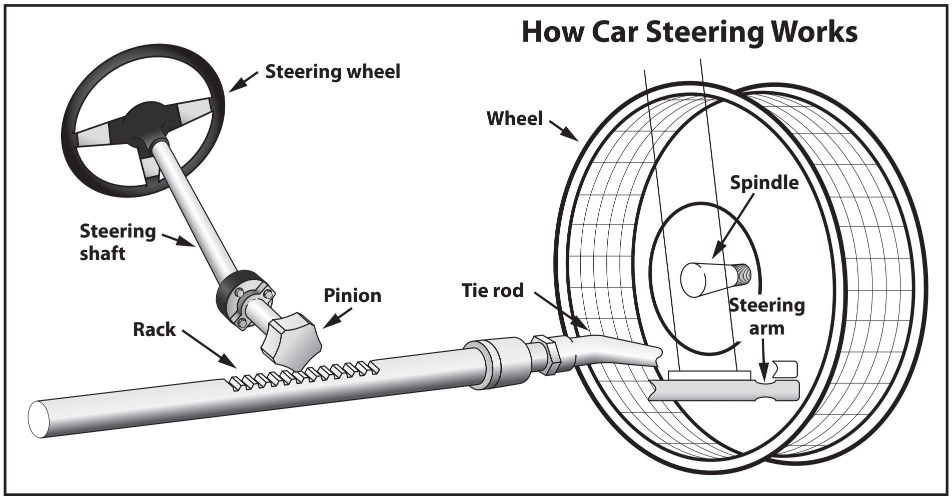
Find the (x, y) coordinates of radar blips located by the following (*distance, angle*) data.

a. (30 km, 40°) **b.** (20 km, 160°)

c. (70 km, 210°) **d.** (15 km, 230°)

e. (45 km, 270°) **f.** (40 km, 310°)

7 A rack-and-pinion gearset is used in the steering mechanism of most cars. The gearset converts the rotational motion of the steering wheel into the linear motion needed to turn the wheels, while providing a gear reduction that makes it easier to turn the wheels. As shown in the sketch below, when you turn the steering wheel, the gear (pinion) spins, moving the rack. The tie rod at the end of the rack connects to the steering arm on the spindle of each wheel.



a. The steering ratio is the ratio of how far you turn the steering wheel to how far the wheels turn. For example, a steering ratio of 18:1 means you must turn the steering wheel 18° in order to turn the front wheels 1°. A higher steering ratio such as 20:1 means you have to turn the steering wheel more to get the wheels to turn through a 1° angle, but less effort is required because of "gearing down." Would you expect lighter, sportier cars to have higher or lower steering ratios than larger cars and trucks? Explain.

b. Suppose the steering ratio of a car is 18:1.

- **i.** Through how many degrees will the front wheels turn if the steering wheel is turned a complete revolution?
- **ii.** Suppose the maximum turn of the front wheels is 75°. How many revolutions of the steering wheel are needed to make the maximum turn?

8 A circle of radius r centimeters has a circumference of $2\pi r$ centimeters.

a. Suppose a point A on the circle rotates through an angle of p radians. What is the length of the arc traversed by the point?

b. Suppose a point A on the circle rotates at p radians per minute. Find the linear velocity of the point.

c. Suppose a circle with radius 10 cm has an angular velocity of 80 radians per second. Find the linear velocity of a point A on the circle.

d. Suppose a point on a circle with radius 10 cm has linear velocity of 30π cm per second. Find the angular velocity of the point.

e. Explain how to convert an angular velocity v (in radians per second) for a circle of radius r centimeters into the linear velocity of a point on the circle.

9 Degree measure of angles is more familiar to you than radian measure. But the difference between the two is just a matter of scale. That is, $1° = \frac{\pi}{180}$, or about 0.0175 radians, and 1 radian $= \left(\frac{180}{\pi}\right)^{\circ}$, or about 57.3°. The following tasks may help you better understand radians.

a. Each of the following is the radian measure of an angle in standard position. Give the quadrant in which the terminal side of each angle lies.

i. 5 **ii.** -5

iii. $-\frac{4\pi}{3}$ **iv.** $\frac{7\pi}{3}$

b. The measure of an angle in standard position is p radians, and the angle's terminal side is in the second quadrant. In what quadrant is the terminal side of an angle with each of the following radian measures? Give all possibilities and explain.

i. $p + 2\pi$ **ii.** $p - \frac{\pi}{2}$

iii. $p + 9\pi$ **iv.** $p - 2$

10 Many variations on the sine and cosine functions are also periodic functions.

a. Find the amplitude, period, and y-displacement for each of the following functions.

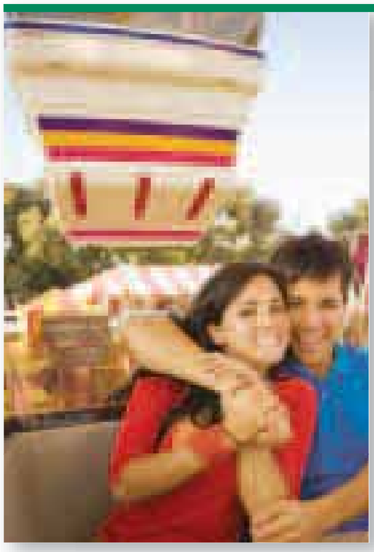
i. $y = 2 \cos(-x) + 3$

ii. $y = -3 \sin 0.1x + 5$

iii. $y = 12 \sin 3x - 8$

b. Suppose data from a periodic variable are fit well by a function of the form $y = a \cos bx + c$ where $a < 0$ and $b > 0$. A plot of the data suggests a y-displacement of -5, amplitude of 7, and period of 6π. Write a function rule for this data.

11 The center of a Ferris wheel in an amusement park is 7 meters above the ground and the Ferris wheel itself is 12 meters in diameter. The wheel turns counterclockwise at a constant rate and takes 20 seconds to make one complete revolution.



a. Yolanda and her friend enter their seat when it is directly below the wheel's center. Sketch a graph that you would expect to show their height above the ground during one minute at full rotational speed, starting from the entry point.

b. Make a table of values for the radian measure t and the height $h(t)$ of Yolanda's seat above the ground at 5 second intervals for the one-minute ride.

c. Use the data pairs in Part b to sketch a graph of the height $h(t)$ as a function of time t. Compare this graph to the one you drew by hand in Part a?

d. Is this function periodic? If so, what is its period? What is its amplitude?

e. Several functions that students predicted would fit the data in Part b are given below. In each case, $\frac{\pi}{10}$ is angular velocity in radians per second, t is time in seconds, and $h(t)$ is height in meters. Determine if any are good fits for the data.

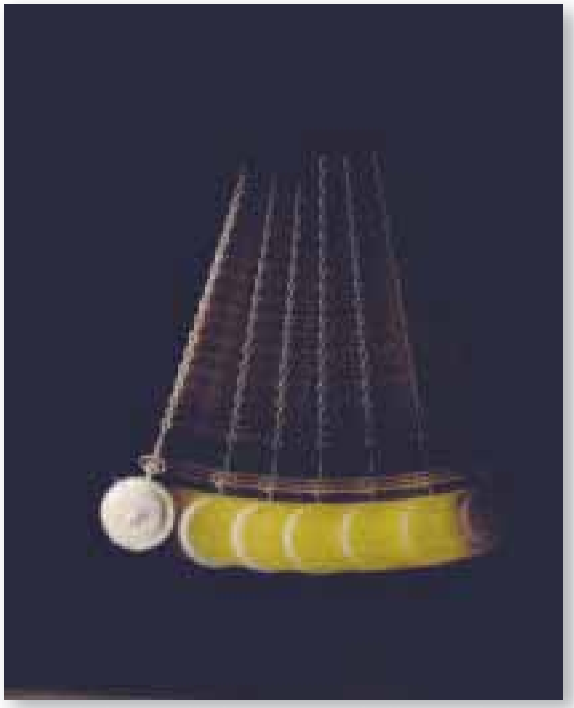
i. $h(t) = 6 \sin \frac{\pi t}{10} + 7$

ii. $h(t) = -6 \cos \frac{\pi t}{10} + 7$

iii. $h(t) = -6 \sin \frac{\pi}{10}(t + 5) + 7$

iv. $h(t) = 6 \cos \frac{\pi t}{10} + 7$

12 Suppose that you are trying to model the motion of a clock pendulum that moves as far as 5 inches to the right of vertical and swings with a period of 2 seconds.



a. Find variations of $d(t) = \cos t$ that fit the conditions for each part below.

i. A modeling function whose values range from -5 to 5 and has a period of 2π

ii. A modeling function that has a period of 2 and whose values range from -1 to 1

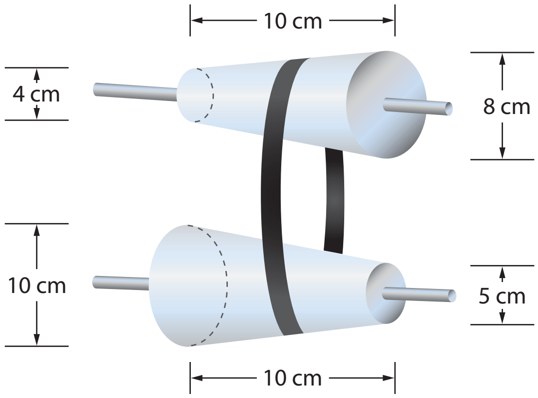
iii. A modeling function that has a period of 2 and whose values range from -5 to 5

b. How are the numbers in the function for Part aiii related to the motion of the pendulum you are modeling?

c. Graph the function that models the motion of the clock pendulum. Identify the coordinates of the t-intercepts and minimum and maximum points of the graph.

Connections

13 Some automobile manufacturers are beginning to use an automatic, continuously-variable transmission (CVT) based on segments of cones. A simplified model is shown below. It consists of two 10-centimeter segments of right cones. The diameters of the circular ends are given. These partial cones form the basis for a *variable-drive* system, in which either partial cone can be moved laterally (left and right) along a shaft.



a. When two pulleys are connected by a chain or belt, their angular velocities are related. The number by which the angular velocity of the driver pulley is multiplied to get the angular velocity of the follower pulley is called the **transmission factor** from driver to follower. For example, in the figure above, the transmission factor for the top cone (driver) to the bottom cone (follower) is $\frac{6}{7.5} = 0.8$. Explain why.

b. If the upper shape is the driver, what are the maximum and minimum transmission factors?

c. Suppose the belt is halfway between the two circular ends of the upper shape. If the lower shape is permitted to move laterally, what range of transmission factors is possible?

d. Describe a position of the cones for which the transmission factor is 1.

14 In Connections Task 13, you considered transmission factors. The transmission factor for pulley A to pulley B can be denoted in function notation as **tf(*AB*)**. Assume A has radius r_1 and B has radius r_2.

a. What does tf(BA) represent?

b. Express tf(AB) in terms of the radii of A and B. Similarly, express tf(BA).

c. If the circumference of A is C_1 and the circumference of B is C_2, express tf(AB) in terms of C_1 and C_2.

d. Using the formula for the circumference of a circle, rewrite your expression in Part c as one involving only radii (r_1 or r_2). Is this result consistent with Part b? Why or why not?

e. Suppose B turns through an angle of $b°$ whenever A turns through an angle of $a°$. Express tf(AB) in terms of a and b.

15 **Rotation Matrices** In earlier study of coordinate geometry, you discovered that results of several special rotations can be produced by matrix multiplication. For example, for any point $P(x, y)$, the matrix product $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$. So, the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ represents a rotation of 180° or half-turn about the origin. In general, multiplication by a matrix in the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ represents a counterclockwise rotation of θ° about the origin.

a. How does the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ for a rotation of 180° fit the general form for rotation matrices?

b. Write the matrix for a counterclockwise rotation of 30°. Then use matrix multiplication to find the images of the following special points on the unit circle and use geometric reasoning to prove that the matrix multiplication has produced the correct rotation images of the points.

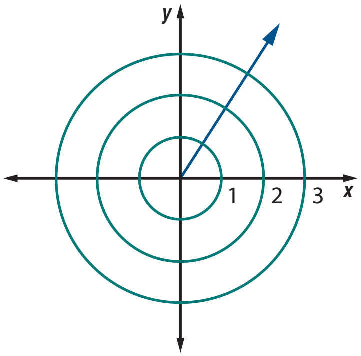
i. (1, 0) **ii.** $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **iii.** (0, 1)

16 An angle of 1 radian is shown in standard position below.

a. What is the length of the arc in the circle of radius 1 traversed by rotating through 1 radian?

b. What transformation takes the circle of radius 1 to that of radius 2? What transformation takes the circle of radius 3 to the circle of radius 2?

c. What are the lengths of the arcs in the circles of radii 2 and 3 traversed by rotating through an angle of 1 radian? Explain why this answer makes sense using the transformations in Part b.



17 In Course 2 Unit 7, *Trigonometric Methods*, you used properties of special right triangles to determine the trigonometric ratios for angles of 30°, 45°, and 60°. You also found the values of these ratios for 0° and 90°.

a. Complete the table at the right giving the sine and cosine of the radian equivalent of each of these angles.

b. In this lesson, you learned that the sine and cosine functions can be used to model the motion of a circle with radius 1 centered at the origin. Sketch this circle and mark the points that lie on the terminal side of each of the five angles in the table. Write the coordinates of each point.

t (in radians)	sin t	cos t
0		
	$\frac{1}{2}$	
$\frac{\pi}{4}$		$\frac{\sqrt{2}}{2}$
	$\frac{\sqrt{3}}{2}$	
$\frac{\pi}{2}$		

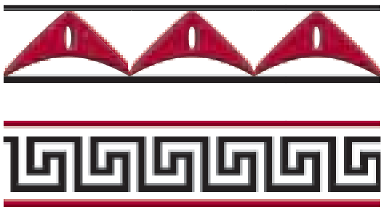
c. The terminal sides of angles measuring 0 and $\frac{\pi}{2}$ radians each lie on one of the coordinate axes. There are two other angles between 0 and 2π radians whose terminal sides lie on an axis. What are the radian measures of these angles? What are the sine and cosine of each angle?

d. In Part b, you used geometric reasoning to calculate cosine and sine of three angles with terminal side in the first quadrant. Use those results and symmetry of the circle to find the cosine and sine of nine more angles—three with terminal side in Quadrant II, three with terminal side in Quadrant III, and three with terminal side in Quadrant IV.

e. In which quadrants are the sines of angles positive? Negative? Answer the same questions for the cosine.

18 Strip or frieze patterns are used in architecture and interior design. The portions of frieze patterns below are from the artwork on pottery of the San Ildefonso Pueblo, New Mexico.

a. What aspects of the graphs of the sine and cosine functions are similar to these strip patterns?



b. What is the shortest nonzero distance you could translate the graph of $y = \sin x$ so that it would map onto itself? Answer the same question about the graph of $y = \cos x$.

c. What symmetries do the graphs of $y = \sin x$ and $y = \cos x$ have? Describe any line of reflection or center and angle of rotation involved in each symmetry.

19 Use the language of geometric transformations to describe the relationships between the graphs of the following pairs of functions.

a. $-\cos t$ and $\cos t$

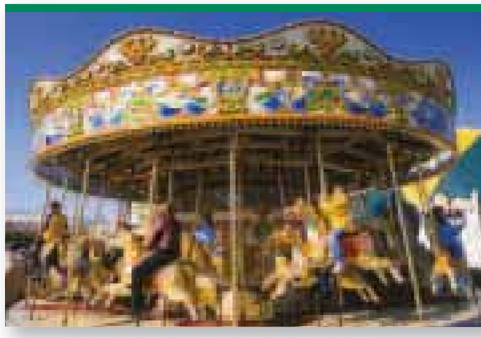
b. $5 + \sin t$ and $\sin t$

c. $5 \cos t$ and $\cos t$

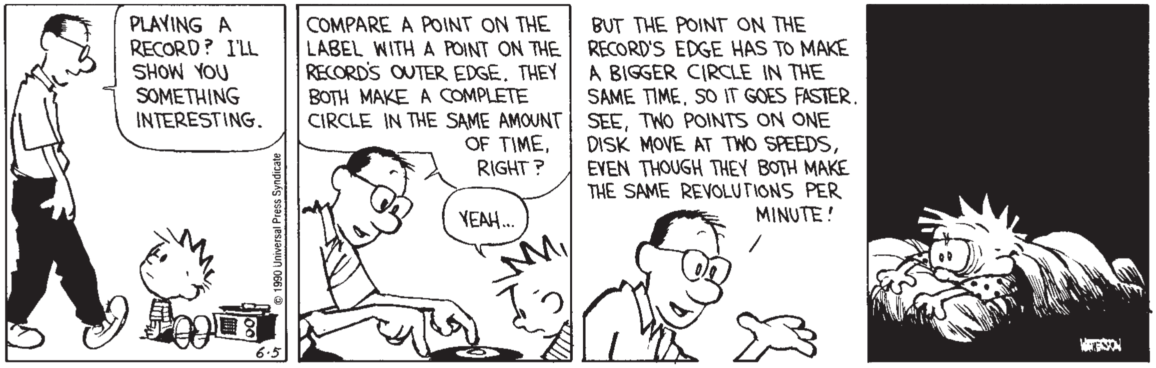
Reflections

20 A popular amusement park ride for young children is the carousel or merry-go-round. A typical carousel consists of a circular platform with three circles of horses to ride, an inner circle, a middle circle, and an outer circle.

Children and sometimes adults ride these horses as the carousel goes round and round. In which circle should a child ride for the slowest ride? For the fastest ride? Why?



21 Read the cartoon below. Using mathematical ideas that you developed in Investigation 1, write a paragraph explaining to Calvin how two points on a record can move at two different speeds.



22 Karen wishes to evaluate sin 25° using her calculator. She presses SIN 25) ENTER. The calculator displays the screen below. What indicates that there must be an error? What do you think the error is?

23 According to Morris Kline, a former Professor of Mathematics at New York University, "The advantage of radians over degrees is simply that it is a more convenient unit. … The point involved here is no different from measuring a mile in yards instead of inches."

Why do you think Kline believes the radian is a "more convenient unit"? (Source: Morris Kline, *Mathematics for Liberal Arts*, Addison Wesley, 1967. page 423)

24 If you have ever ridden a Ferris wheel that was rotating at a constant rate, you know that at some positions on the wheel, you feel like you are falling or rising faster than at other points on the wheel. As the wheel rotates counterclockwise, your directed distance to the horizontal axis through the center of the wheel is $r \sin \theta$, where r is the radius of the Ferris wheel. Examine the graph of the sine function.

a. Near what values of θ in radians is the sine function increasing most rapidly? Decreasing most rapidly?

b. Where on the wheel are you located when the sine function is increasing most rapidly? Decreasing most rapidly?

c. Explain what your answers in Parts a and b may have to do with your feelings of falling or rising when you ride a Ferris wheel.

25 In previous courses, you have studied various functions—linear, exponential, power, polynomial, and trigonometric. It is helpful to think about each "family" of functions in terms of basic symbolic rules and their corresponding graphs.

a. Make a general sketch of the graph of each function below. Label the x- and y-intercepts. In each case, assume $a > 1$.

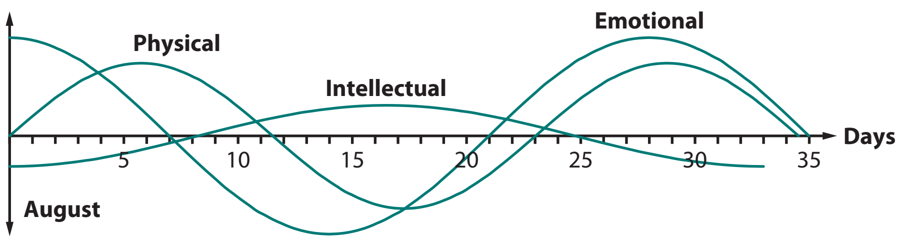
i. $y = ax$ **ii.** $y = a^x$ **iii.** $y = ax^2$

iv. $y = ax^3$ **v.** $y = \frac{a}{x}$ **vi.** $y = \frac{a}{x^2}$

vii. $y = a \sin x$ **viii.** $y = a \cos x$

b. If you are given a graph, what characteristics would you look for to identify the graph as a member of each family in Part a?

26 The *biorhythm* theory asserts that a person's biological functioning is controlled by three inner rhythms which begin at birth: physical, emotional, and intellectual. These rhythms vary sinusoidally with time. Biorhythm graphs are used by athletes as well as industrial firms to predict potential "good" or "bad" performance days for a person. Use the labeled graphs below to determine the period of each biorhythm.



a. Physical **b.** Emotional **c.** Intellectual

d. According to biorhythm theory, when a cycle is near a high point, a person can perform well in an activity requiring the corresponding biological functioning. Similarly, low points in the cycle are associated with times of low performance. What date(s) would be best for a person to run a 10-km race?

Extensions

27 When two rotating circles are linked in a driver/follower relationship, the **transmission factor** from driver to follower can be expressed as the ratio of angular velocities $\frac{v_f}{v_d}$. One model of a 21-speed mountain bike has a pedal crankset of 3 sprockets with 48, 40, and 30 teeth; the bike has 7 sprockets for the rear wheel with 30, 27, 24, 21, 18, and 12 teeth.

a. What is the largest transmission factor available for this bicycle? What is the smallest?

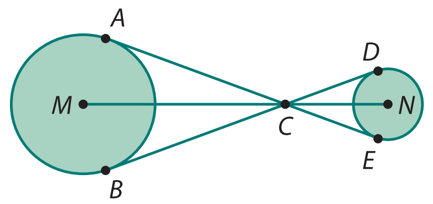
b. Suppose on a cross-country ride, you can maintain an angular velocity of 70 rpm for the pedal sprocket. What is the fastest that you can make the rear wheel turn?

c. The radius of a mountain bike tire is about 33 cm.

i. What is the linear velocity of the rear wheel when the pedal sprocket turns at 70 rpm and the transmission factor is greatest possible?

ii. What is the linear velocity of the rear wheel when the pedal sprocket turns at 70 rpm and the transmission factor is least?

28 In the figure at the right, $\overline{AE}$ and $\overline{BD}$ are tangent to both circles. The point of intersection of the two tangents lies on the line containing the centers of the circles.



a. Draw the radii to the points of tangency in each circle. The result should be four triangles that are similar to one another. Explain why?

b. Suppose the radius of the larger circle is 8 cm, the radius of the smaller circle is 4 cm, and the distance between the centers is 30 cm. Use similar triangles to determine the length of a common tangent segment.

c. Suppose the circles represent pulleys with a belt going around the pulleys as indicated by the common tangents. Given the measures in Part c, determine the length of the belt.

29 In this task, you will explore a shape that is different in form from a circle but can serve a similar design function.

a. Make a cardboard model of this shape as follows.

Step 1. Construct an equilateral triangle of side length 6 cm.

Step 2. At each vertex, place a compass and draw the arc of a circle with radius 6 cm that gives the two other vertices. When all three pairs of vertices have been joined by these arcs, you have a rounded triangle-like figure, known as a *Reuleaux triangle* (pronounced re-low).

Step 3. Carefully cut out the Reuleaux triangle.

b. Draw a pair of parallel lines 6 cm apart, and place the edge of a ruler along one of the lines. Place your cardboard Reuleaux triangle between the lines and roll it along the edge of the ruler. Note any unusual occurrences.

c. Conduct library or Internet research on the Wankel engine. How is it related to your model?

d. Investigate the concept of *shapes of constant width*. What are two other examples of such shapes?

30 An important idea in calculus involves the ratio $\frac{\sin x}{x}$ for values of x close to 0. Make a table with columns headed x and $\frac{\sin x}{x}$.

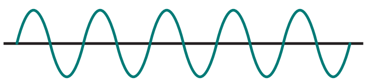
a. Using radian mode, complete one table for $x = 1, 0.5, 0.25, 0.1, 0.01, 0.001$, and 0.0001. What appears to be happening to $\frac{\sin x}{x}$ as x approaches 0 in radian mode?

b. Using degree mode, complete a second table with the same headings and same values of x. What appears to be happening to $\frac{\sin x}{x}$ as x approaches 0 in degree mode?

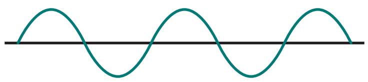
c. Compare your results in Parts a and b to the difference in scale of degrees and radians. Do you see any connection?

31 The electronic pattern of simple sounds, such as those you can view on an oscilloscope, are modeled by trigonometric functions of the form $y = a \sin bt$ or $y = a \cos bt$, where b is expressed in radians per second and t is time in seconds. The period of a sound wave relates to the frequency or number of waves that pass a point each second. The greater the frequency (or shorter the period) of the sound wave, the higher the pitch of the sound.

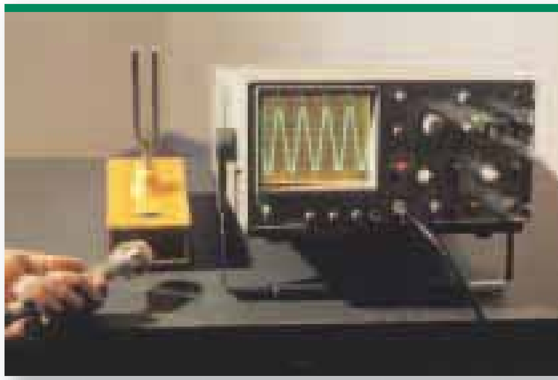
High-frequency Sound Wave



Low-frequency Sound Wave



The amplitude $|a|$ represents loudness. The greater the amplitude of the wave, the louder the sound. A good example of a simple sound is the sound produced by a piano tuning fork.



a. The middle C tuning fork oscillates at 264 cycles per second. This is called its frequency. How many radians per second is this?

b. Write a function rule that models the sound of middle C when a is 1.

c. What is the period of the function? How is it related to the frequency of 264 cycles per second?

d. Graph four cycles of the function. If you use 0 for the minimum x value, what should you use for the maximum value?

e. The C note two octaves above middle C has a frequency of 1,056 cycles per second. Model this sound with a function rule. What is the period?

f. The C note that is one octave below middle C has a frequency of 132 cycles per second. Model this sound with a function rule. Graph the function in the same viewing window with the middle C graph from Part b. What do you think is the frequency of the C note one octave above middle C?

32 In this lesson, you learned that the distance from a point on a circle of radius r to the vertical axis of the circle is given by $y = r \cos x$. Similarly, the distance to the horizontal axis of the circle is $y = r \sin x$.

Another way to describe circular motion is to use *parametric equations*, which you will study more fully in Course 4. With parametric equations, a third variable t is introduced, and the x- and y-coordinates of points on the circle are written as separate functions of t.

For an introductory look at this idea for a circle of radius 5 centimeters, set your calculator or computer software to parametric mode and use degrees. Set the viewing window so that t varies from 0 to 360 in steps of 5, x varies from −9 to 9, and y varies from −6 to 6. Then in the functions list, enter $X_T = 5 \cos t$ and $Y_T = 5 \sin t$ as the first pair of functions.

a. Graph and trace values for $t = 30, 60, 90$, and so on.

b. What physical quantity does X_T represent? What physical quantity does Y_T represent?

c. What measurement does t represent? What are the units of t?

d. How much does t change in one revolution of the circle?

e. If one revolution of the circle takes 20 seconds, what is the angular velocity of a point on the circle in units of t per second?

f. What is the linear velocity of a point on the circle in meters per second?

g. What parametric equations would represent circular motion on a circle with a radius of 3 centimeters? Enter them as the second pair of parametric equations in your calculator's function list.

h. Graph at the same time the equations for circles with radii of 3 and 5 cm. (On some calculators, you need to set the mode for simultaneous graphs.) Watch carefully.

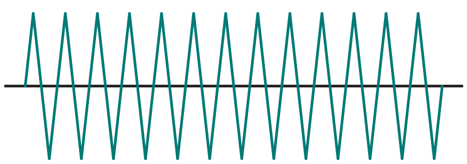
- **i.** If the circles both turn with the same angular velocity, on which circle does a point have the greater linear velocity?
- **ii.** What is the linear velocity of the slower point?

33 AM and FM radio stations broadcast programs at various frequencies. The frequency f of a periodic function is the number of cycles per unit of time t. Radio signals broadcast by a given station transmit sound at frequencies that are multiples of 10^3 cycles per second (kilohertz, kHz) or 10^6 cycles per second (megahertz, MHz).

AM radio stations broadcast at frequencies between 520 and 1,610 kHz. An AM radio station broadcasting at 675 kHz sends out a **carrier signal** whose function rule and graph are of the form shown in the diagram below.

Carrier Wave

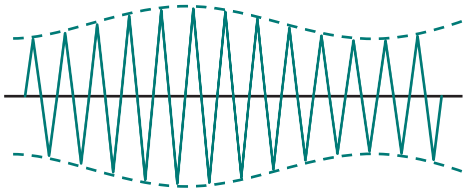
$y = A_0 \sin (2\pi f \cdot t)$



To transmit program sounds of varying frequencies, an AM station varies or "modulates" the amplitude of the carrier signal. The initials AM refer to **amplitude modulation**. An amplitude-modulated signal has a function rule and graph of the form shown below. Note that the period, and therefore, the frequency, of the carrier wave is left unchanged. The variable amplitude $A_0(t)$ is itself a sinusoidal function representing the broadcast programming.

AM Wave

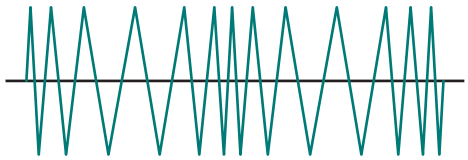
$y = A_0(t) \sin (2\pi f \cdot t)$



FM radio stations broadcast at frequencies between 87.8 and 108 MHz. An FM (**frequency modulation**) station transmits program sounds by varying the frequency of the carrier signal while keeping the amplitude constant. A frequency-modulated signal has a function rule and graph of the form shown below. The variable frequency $f(t)$ is itself a sinusoidal function representing the broadcast programming.

FM Wave

$y = A_0 \sin 2\pi f(t)t$



a. A radio station operates at a frequency of 98.5 MHz.

  i. Is the station AM or FM?

  ii. What position on your radio dial would you tune to receive the signals broadcast by this station?

b. Write a function rule for the carrier signal of a radio station that broadcasts at a frequency of 610 kHz. What is the period of the carrier wave for this station?

c. Write a function rule in terms of its variable frequency $f(t)$ for the carrier signal of a radio station that broadcasts at a frequency of 104.1 MHz. What is the period of the carrier wave for this station?

Review

34 Use what you know about the relationships between different units of measurement to complete the following.

a. 7.5 ft = _____ in. = _____ yd

b. 1,780 mm = _____ cm = _____ m

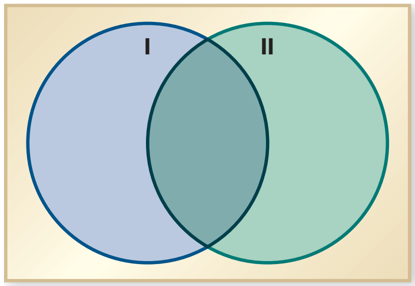
c. 1 yd^2 = _____ ft^2 = _____ in^2

35 Use algebraic reasoning to find the coordinates of the maximum or minimum point, the x-intercept(s), and the y-intercept of each function.

a. $f(x) = -(x - 5)^2 + 7$

b. $g(x) = x^2 + 6x - 1$

36 Use properties of quadrilaterals to place the special types of quadrilaterals (parallelogram, rectangle, square, rhombus, kite, trapezoid, isosceles trapezoid) in the correct region of Venn diagrams similar to the one below.



a. **I** Both pairs of opposite sides are parallel.

II Has at least one line of symmetry.

b. **I** Each diagonal divides the figure into two congruent triangles.

II Both diagonals are the same length.

37 Solve each equation for the indicated variable.

a. $C = 2\pi r$ for r

b. $A = \frac{1}{2}h(b_1 + b_2)$ for b_2

c. $E = mc^2$ for c

d. $d = \frac{m}{v}$ for v

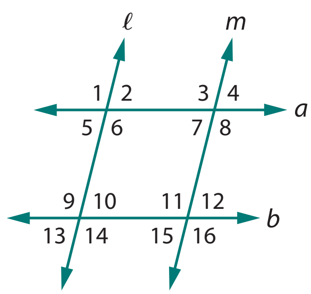
38 Suppose that a machine fills bottles of water so that the distribution of the amounts of water in the bottles is normal, with a mean of 16 fluid ounces and a standard deviation of 0.1 ounces.

a. Draw a sketch of the amount of water in the bottles. Include a scale on the horizontal axis.

b. What percentage of the bottles will have less than 16.2 ounces of water in them?

c. What percentage of the bottles will have at least 15.9 ounces of water in them?

39 Using the diagram at the right, determine, for each set of conditions, whether the conditions imply that any pairs of lines are parallel. If so, indicate which lines are parallel.



a. m∠9 = m∠11

b. m∠13 = m∠7

c. m∠3 = m∠16

d. m∠1 + m∠12 = 180° and m∠9 = m∠11

40 Simplify each algebraic fraction as much as possible.

a. $\dfrac{16x + 12}{3x + 15}$

b. $\dfrac{x^2 - 9x}{x^2 - 81}$

c. $\dfrac{x^2 + 8x + 15}{x^2 + 2x - 3}$

d. $\dfrac{x - 1}{4x - 4}$

41 Use each *NOW-NEXT* rule to produce a table of values that illustrates the pattern of change from the start value through 5 stages of change.

a. $NEXT = NOW + 2.5$, starting at 3

b. $NEXT = \frac{1}{2}NOW - 50$, starting at 450

42 The number of bacteria in a wound can be modeled using the function $N(t) = 3(8^t)$, where t represents the number of hours since the wound occurred.

a. Evaluate and explain the meaning of $N(2)$.

b. How many bacteria were introduced into the wound when it occurred?

c. How long does it take for the number of bacteria in the wound to double?

d. Write a *NOW-NEXT* rule that could be used to find the number of bacteria present after any number N of complete hours.

43 Between July 1, 2006, and July 1, 2007, the fastest growing metropolitan area in the United States was Palm Coast, Florida. It had an annual growth rate of 7.2%. During that same time period, the Youngstown, Ohio, area saw a decrease in population of 1%. (Source: www.census.gov/Press-Release/www/releases/archives/population/009865.html)

a. The population of Palm Coast was 82,433 on July 1, 2006. What was its approximate population on July 1, 2007?

b. The population of the Youngstown, Ohio, area was 576,602 on July 1, 2006. What was its approximate population on July 1, 2007?

c. Which of these *NOW-NEXT* rules could be used to help calculate the population of the Youngstown, Ohio, area in future years if this rate of decline continues? All rules have a starting value of 576,602.

- $NEXT = 0.01NOW$
- $NEXT = 0.99NOW$
- $NEXT = NOW - 0.01NOW$
- $NEXT = NOW + 0.01NOW$

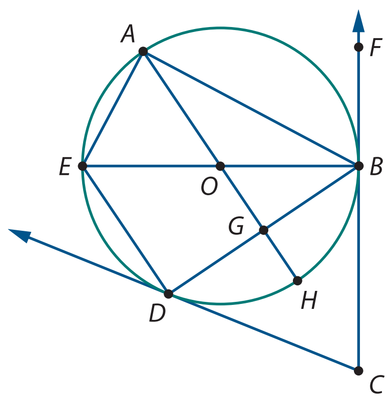
LESSON 3

Looking Back

In this unit, you explored and proved important properties of special lines and angles in circles. To reason about these properties, you drew on methods that used congruent triangles, symmetry, coordinate geometry, and trigonometry. You then built on these circle properties by embarking on a study of characteristics and applications of revolving circles. Your study of circular motion began with a study of pulleys and gears, moved to linear and angular velocity, introduced radian angle measure, and culminated with the study of the sine and cosine functions and their role in modeling aspects of circular motion and other periodic phenomena.

The following tasks will help you review, pull together, and apply what you have learned.

1 A circle with center O has diameters $\overline{BE}$ and $\overline{AH}$. Rays $\overrightarrow{CB}$ and $\overrightarrow{CD}$ are tangent to the circle at points B and D, respectively, and G is the midpoint of chord $\overline{BD}$.



a. Explain why $m\angle EAB = m\angle EBC = m\angle AGB$.

b. Suppose $m\widehat{AED} = 124°$ and $m\angle ABF = 62°$. Find the degree measures of $\angle AED$, $\angle BCD$, and $\widehat{DE}$. Explain your reasoning.

2 For each of the following statements, prove the statement if it is true or give a counterexample if it is false.

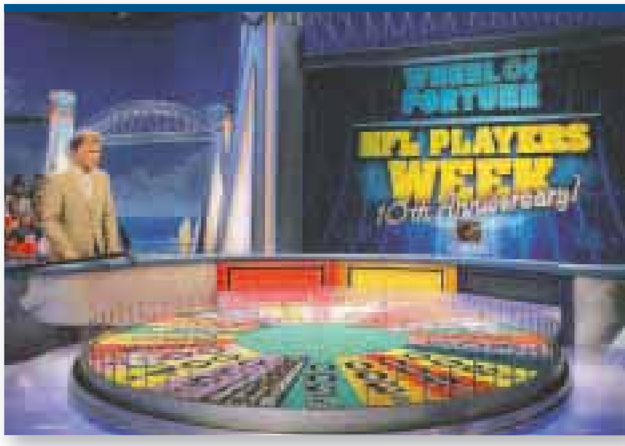
a. The midpoint of the common chord of two circles that intersect in two points lies on the line through the centers of the circles.

b. If inscribed angle $\angle ABC$ in a circle with center O measures 120°, then central angle $\angle AOC$ measures 120°.

c. The angle bisector of any angle inscribed in a circle contains the center of the circle.

d. If the line through the midpoints of two chords of a circle contains the center of the circle, then the chords are parallel.

3 *Wheel of Fortune* has been the most popular game show in the history of television. On the show, three contestants take turns spinning a large wheel similar to the one at the right. The result determines how much that contestant wins as she or he progresses toward solving a word, phrase, or name puzzle. The wheel is divided into 24 sectors of equal size, each corresponding to a dollar value or some other outcome.



a. If the wheel spins through 3.8 counterclockwise revolutions, what is the degree measure of the angle through which the wheel spins?

b. What is the measure of the minimum positive angle that is *coterminal* with (has the same terminal side as) the angle in Part a?

c. Suppose the wheel starts at the center of the “Bankrupt” sector and spins through 1,055°. Will it stop at “Bankrupt”? Describe the intervals of angle measures that will return the wheel to the “Bankrupt” sector.

Imagine a coordinate system superimposed on the wheel of fortune with its origin at the center of the wheel. Suppose a contestant spins the wheel releasing it at the “3 o’clock” position on the coordinate system, and the wheel completes 2.8 revolutions in 20 seconds.

d. What is the angular velocity in revolutions per second? In radians per second?

e. If the wheel is 10 feet in diameter, what is the average linear velocity in feet per second of a point on the edge of the wheel?

f. What are the coordinates of the release point when the wheel stops at the end of 20 seconds? Explain your reasoning.

4 The Thames River passes through the heart of London, England. Ships entering that part of the river need to pass under a number of bridges like the famous Tower Bridge. The height of the Tower Bridge above the river (in meters) varies over time (in hours following high tide) according to $d(t) = 12 + 3.4 \cos 0.5t$.

a. What are the maximum and minimum distances from the bridge to the river?

b. What are the period and amplitude of variation in distance from the bridge to the river?

c. How often in one day does the distance from bridge to river complete a full cycle from maximum to minimum and back to maximum?

Summarize the Mathematics

In this unit, you described and reasoned about important properties of special lines, segments, and angles that intersect circles. You also analyzed circular motion including linear and angular velocity, radian measure of angles, and circular functions.

a. What is the relationship between a line k tangent to a circle with center O at point A and radius $\overline{OA}$?

b. What is the relationship between the measure of a central angle in a circle and the measure of any inscribed angle that intercepts the same arc of the circle?

c. If a radius of a circle intersects a chord of that circle at its midpoint, what are the measures of the angles formed?

d. How is the angular velocity of one rotating pulley (radius r_1) related to the angular velocity of a second pulley (radius r_2) connected to it if:

   i. $r_1 = r_2$ **ii.** $r_1 > r_2$ **iii.** $r_1 < r_2$

e. What does it mean to say that an angle or rotation has radian measure t? What would be the approximate corresponding measure in degrees?

f. If $P(r, 0)$ is a point on a circle of radius r centered at the origin of a coordinate system, what are the coordinates of the image of P after a counterclockwise rotation of θ about the origin?

g. What do the period, amplitude, and frequency of a periodic function tell about the pattern of values and graph of the function?

h. When $f(t) = a \sin bt + c$ or $g(t) = a \cos bt + c$ are used to model the periodic pattern of change in some quantity, what do the values of a, b, and c tell about the situation being modeled?

Be prepared to explain your ideas to the class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.