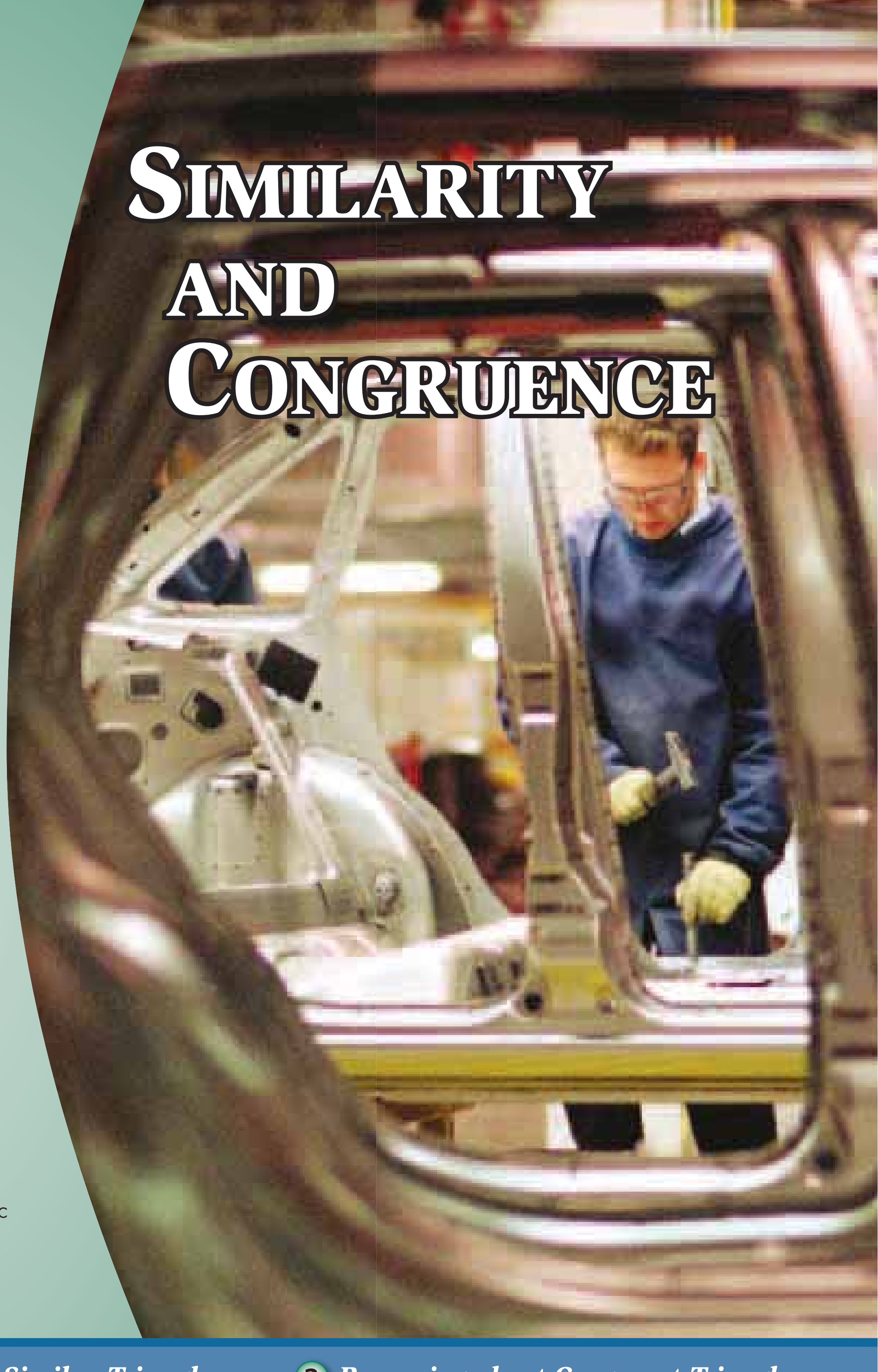
UNIT 3

Similarity and Congruence



Similarity and congruence are key related ideas in design, manufacturing, and repair of products, from cell phones and MP3 players to automobiles and space shuttles. When individual components that comprise these products are manufactured to design plans and specifications, the components are similar in shape to the design images and congruent to each other, whether manufactured at the same or different plants. Because individual components are congruent, they can be easily interchanged in assembly line production or in repair work.

In this unit, you will develop understanding and skill in the use of similarity and congruence relations to solve problems involving shape and size. In each of the two lessons, you will have many opportunities to continue to develop skill and confidence in reasoning deductively using both synthetic and coordinate methods.

Lessons

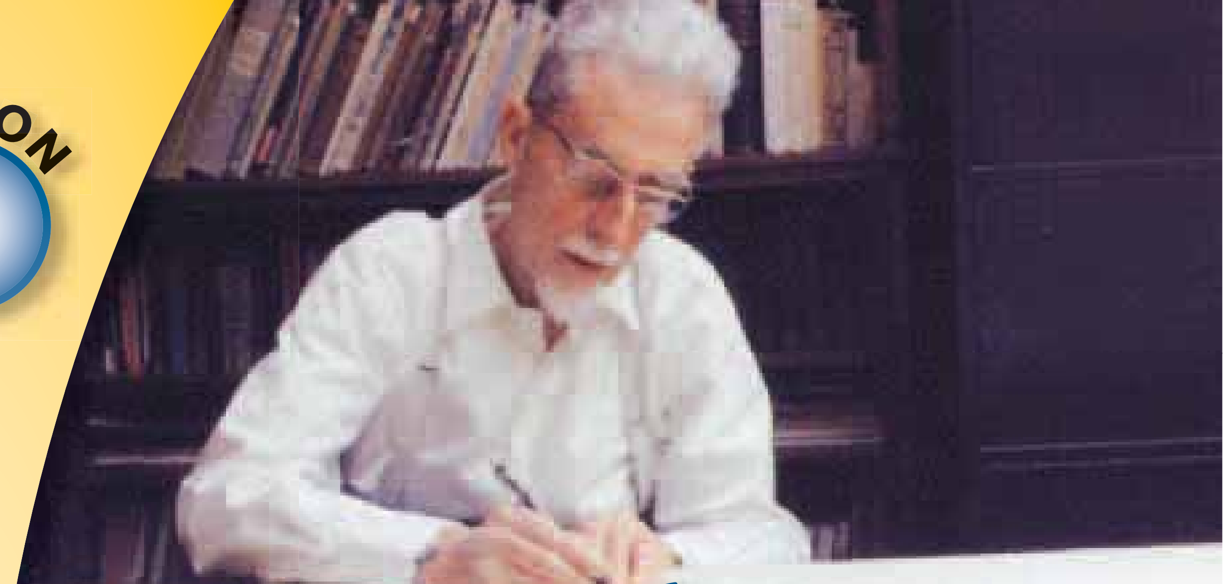
1 *Reasoning about Similar Triangles*

Derive sufficient conditions for similarity of triangles using the Law of Cosines and the Law of Sines. Use similarity conditions to prove properties of triangles and size transformations, and use those conditions and properties to solve applied problems.

2 *Reasoning about Congruent Triangles*

Derive sufficient conditions for congruence of triangles. Use congruence conditions to prove properties of triangles and congruence-preserving transformations, to establish sufficient conditions for quadrilaterals to be (special) parallelograms and for special parallelograms to be congruent, and to solve applied problems.

LESSON 1



Reasoning about Similar Triangles

In Course 1 of *Core-Plus Mathematics*, you discovered combinations of side or angle measures that were sufficient to determine if two triangles were congruent, that is, whether they had the same shape and size. You also explored conditions under which congruent copies of triangles or other polygons could tile a plane.

The mathematically inspired print shown below was created by the Dutch artist M. C. Escher for a book about tilings of a plane. The print provides an interesting twist on use of a basic shape to cover a portion of a plane. It also represents an early attempt by the artist to depict infinity.

Source: *M. C. Escher, His Life and Complete Graphic Works* © 1981

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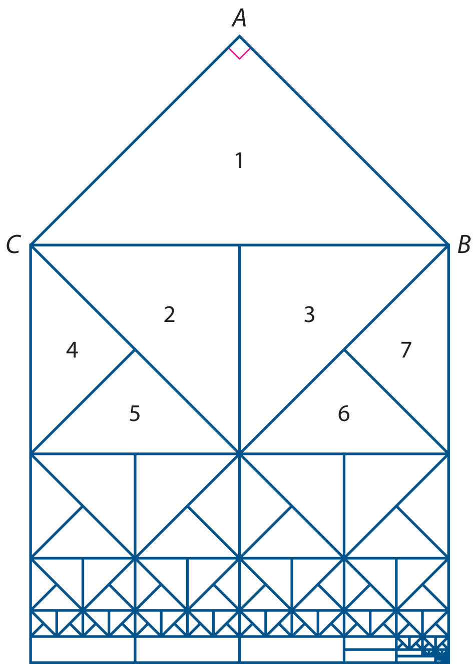
Escher's framework for the design of his print is shown below. Isosceles right triangle *ABC* is the starting point.



Think About This Situation

Study the Escher print and the framework used to create it.

a How are the lizards in the Escher print alike? How are they different? How could you test your ideas?

b Starting with $\triangle ABC$, the framework was created by next drawing right isosceles triangles 2 and 3. How do you think these two triangles were determined? How are triangles 2 and 3 related to each other? To triangle 1? Why?

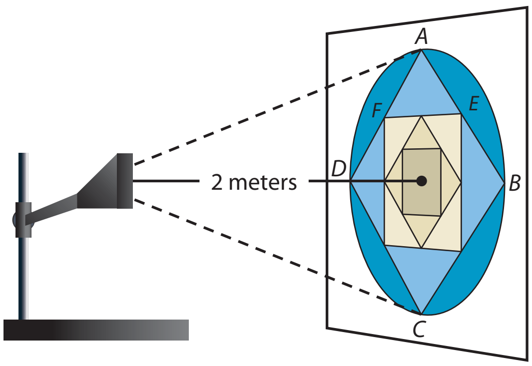
c How do you think triangles 4 and 5 and triangles 6 and 7 were created? How are these pairs of triangles related? How are they related to triangle 1? Explain your reasoning.

d Describe a way to generate the remaining portion of the framework.

In the first lesson of this unit, you will explore similarity of polygons and use deductive reasoning to determine sufficient conditions for similarity of triangles. You will use those relationships to solve applied problems and prove important geometric theorems.

Investigation 1 When Are Two Polygons Similar?

Whenever the light source of a projector is positioned perpendicular to a screen, it enlarges images on the film into *similar* images on the screen. The enlargement factor from the original object to the projected object depends on the distance from the projector to the screen (and special lens features of the projector).



As you work on the problems of this investigation, look for answers to the following questions:

How can you test whether two polygons are similar?

How can you create a polygon similar to a given polygon?

1 Examine the test pattern shown on the screen.

a. How would you test if quadrilateral *ABCD* on the screen is similar to the original quadrilateral that is being projected?

b. How would you test if $\triangle AEF$ on the screen is similar to the original triangle that is being projected?

c. How would you move the projector if you wanted to make the test pattern larger? Smaller?

d. How could you determine how much larger projected $\triangle AEF$ is than the original triangle? Explain what measurements you would make and how you would use them.

e. Compare your answers with other students. Resolve any differences.

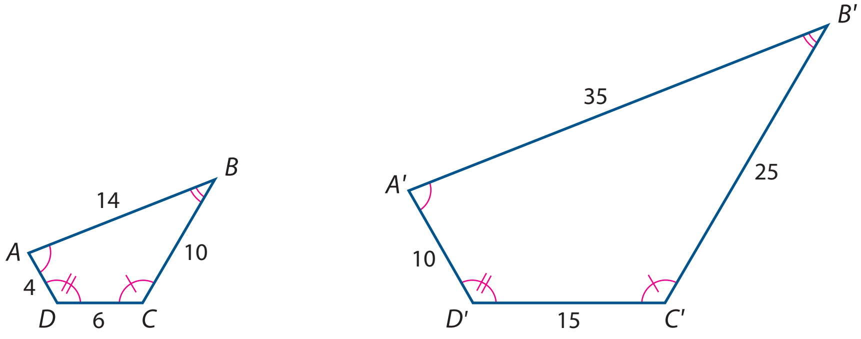
2 Cameras record images digitally and on film. When a camera lens is positioned perpendicular to the plane containing an object, the object and its recorded image are similar. This photograph was taken by a digital camera.

a. Describe how you could use information in the photograph to help determine the actual dimensions of the face of the cell phone.

b. What are those dimensions?

The following problem formalizes some of the ideas you used in Problems 1 and 2.

3 Two polygons with the same number of sides are **similar** provided their corresponding angles have the same measure and the ratios of lengths of corresponding sides is a constant.



In the above diagram, *quadrilateral $A'B'C'D' \sim$ quadrilateral ABCD*. The symbol $\sim$ means "is similar to."

$$m\angle A' = m\angle A,\ m\angle B' = m\angle B,\ m\angle C' = m\angle C, \text{ and } m\angle D' = m\angle D.$$

$$\frac{A'B'}{AB} = \frac{35}{14} = \frac{5}{2} \text{ or equivalently } A'B' = \frac{5}{2}AB.$$
$$\frac{B'C'}{BC} = \frac{25}{10} = \frac{5}{2} \text{ or equivalently } B'C' = \frac{5}{2}BC.$$
$$\frac{C'D'}{CD} = \frac{15}{6} = \frac{5}{2} \text{ or equivalently } C'D' = \frac{5}{2}CD.$$
$$\frac{D'A'}{DA} = \frac{10}{4} = \frac{5}{2} \text{ or equivalently } D'A' = \frac{5}{2}DA.$$

The constant $\frac{5}{2}$ is called the **scale factor** *from quadrilateral ABCD to quadrilateral $A'B'C'D'$*. It scales (multiplies) the length of each side of quadrilateral $ABCD$ to produce the length of the corresponding side of quadrilateral $A'B'C'D'$.

a. What is the scale factor from quadrilateral $A'B'C'D'$ to quadrilateral $ABCD$?

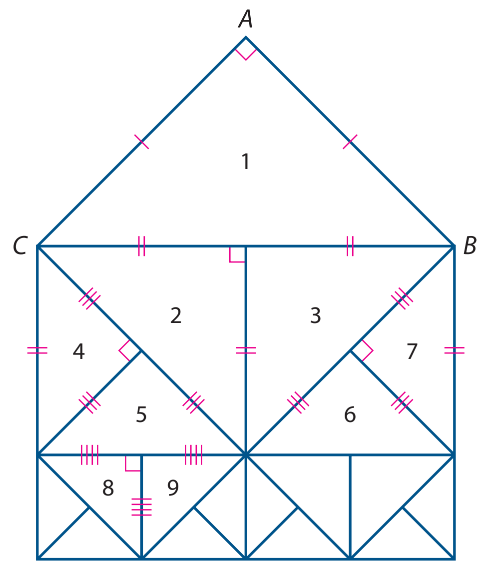
b. How, if at all, would you modify your similarity test in Problem 1 Part b using the above definition of similarity?

c. If two pentagons are similar, describe how to find the scale factor from the smaller pentagon to the larger pentagon. Then describe how to find the scale factor from the larger pentagon to the smaller pentagon.

d. Suppose $\triangle PQR \sim \triangle XYZ$ and the scale factor from $\triangle PQR$ to $\triangle XYZ$ is $\frac{3}{4}$. Write as many mathematical statements as you can about pairs of corresponding angles and about pairs of corresponding sides. Compare your statements with other students. Resolve any differences.

Knowing that two triangles are similar allows you to conclude that the three pairs of corresponding angles are congruent and that the three pairs of corresponding sides are related by the same scale factor. Conversely, if you know that the three pairs of corresponding angles are congruent and the three pairs of corresponding sides are related by the same scale factor, you can conclude that the triangles are similar.

4 The diagram below is a portion of the framework for Escher's print that you examined in the Think About This Situation (page 163). Recall that $\triangle ABC$ is an isosceles right triangle. Assume $BC = 2$ units.



a. Compare the markings on the sides and angles of the triangles with your analysis of the framework. Explain why the markings are correct.

b. Determine if each statement below is correct. If so, explain why and give the scale factor from the first triangle to the second triangle. If the statement is not correct, explain why.

i. $\triangle 1 \sim \triangle 3$ **ii.** $\triangle 2 \sim \triangle 6$

iii. $\triangle 4 \sim \triangle 6$ **iv.** $\triangle 8 \sim \triangle 3$

v. $\triangle 9 \sim \triangle 1$

5 Based on their work in Problem 4, several students at Black River High School made conjectures about families of polygons. Each student tried to outdo the previous student. For each claim, explain as precisely as you can why it is true or give a counterexample.

a. Monisha conjectured that *all isosceles right triangles are similar*.

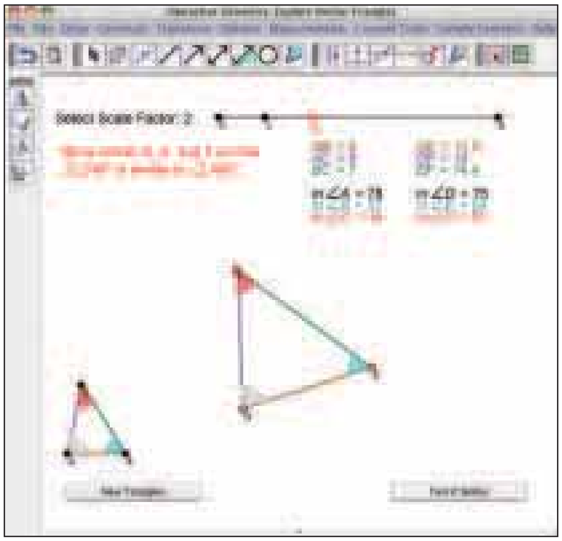
b. Ahmed conjectured that *all equilateral triangles are similar.*

c. Loreen claimed that *all squares are similar.*

d. Jeff conjectured that *all rhombuses are similar.*

e. Amy claimed that *all regular hexagons are similar.*

6 The "Explore Similar Triangles" custom tool is designed to help you construct similar triangles based on a chosen scale factor. As a class or in pairs, experiment with the software. Conduct at least three trials. Observe what happens to the angle and side measures of the triangle as you drag the red vertices.



a. For each trial:

- choose a target scale factor and construct $\triangle DEF$ so that it is similar to $\triangle ABC$.
- use the definition of similarity to justify that your triangles are similar.

b. Think about the process you used to create a triangle similar to $\triangle ABC$.

i. When dragging $\triangle DEF$ to form a triangle similar to $\triangle ABC$, how did you decide how to initially drag the vertices?

- Did you use the angle measurements?
- Did you use the length measurements?

ii. Once you knew that two pairs of corresponding sides were related by the same scale factor, did that guarantee that the third pair was automatically related by the same scale factor?

c. How do you think the software determines if your constructed triangle is similar to the original triangle?

d. How does the software visually demonstrate the similarity of two triangles?

Summarize the Mathematics

In this investigation, you explored similarity of polygons with a focus on testing for similarity of triangles.

a Explain why not all rectangles are similar.

b What is the fewest number of measurements needed to test if two rectangles are similar? Explain.

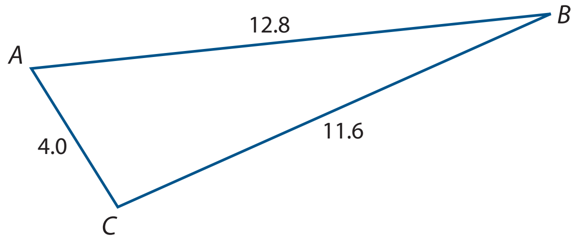
c Explain why any two regular n-gons are similar. How would you determine a scale factor relating the two n-gons?

d What needs to be verified before you can conclude that two triangles, $\triangle PQR$ and $\triangle UVW$, are similar?

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

Each triangle described in the table below is similar to $\triangle ABC$. For each triangle, use this fact and the additional information given to:



a. identify the correspondence between its vertices and those of $\triangle ABC$.

b. determine the remaining table entries.

Triangle Angle Measures			Shortest Side Length	Longest Side Length	Third Side Length	Scale Factor from $\triangle ABC$
$m\angle A = 64°$	$m\angle B = 18°$	$m\angle C = 98°$	$AC = 4.0$	$AB = 12.8$	$BC = 11.6$	1
$m\angle D = ?$	$m\angle E = 64°$	$m\angle F = 18°$				2
$m\angle G = ?$	$m\angle H = ?$	$m\angle I = ?$		$IG = 6.4$	$GH = 5.8$	
$m\angle J = ?$	$m\angle K = 18°$	$m\angle L = 98°$	$JL = 14.0$			

Investigation 2 Sufficient Conditions for Similarity of Triangles

In Investigation 1, you proposed a method by which computer software could test if two triangles were similar. As you tested pairs of triangles, you compared corresponding angle measures and corresponding side lengths. You may have observed that if certain conditions on measures of some corresponding sides or angles are met, similar relationships must then hold for the remaining corresponding parts. In this investigation, you will explore minimal conditions that will ensure that two triangles are similar. You will find the previously proven Law of Sines and Law of Cosines helpful in your work. These two laws are reproduced here for easy reference.

Law of Sines

In any triangle ABC with sides of lengths a, b, and c opposite $\angle A$, $\angle B$, and $\angle C$, respectively, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ or equivalently, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

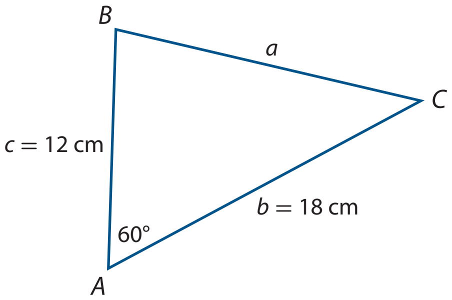
Law of Cosines

In any triangle ABC with sides of lengths a, b, and c opposite $\angle A$, $\angle B$, and $\angle C$, respectively, $c^2 = a^2 + b^2 - 2ab \cos C$.

As you work on the problems of this investigation, look for answers to the following question:

What combinations of side or angle measures are sufficient to determine that two triangles are similar?

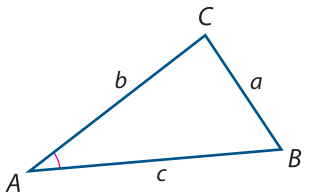
1 To begin, consider $\triangle ABC$ with b, c, and $m\angle A$ given as shown below.



a. Calculate a, $m\angle B$, and $m\angle C$.

b. Are a, $m\angle B$, and $m\angle C$ *uniquely determined*? That is, is there exactly one value possible for each? Explain your reasoning.

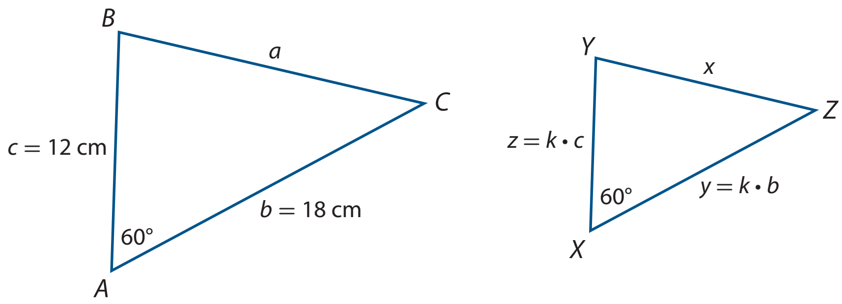
c. In general, if you know values of b, c, and $m\angle A$, can values for a, $m\angle B$, and $m\angle C$ always be found? Could any of a, $m\angle B$, or $m\angle C$ have two or more values when b, c, and $m\angle A$ are given? Explain.



d. Summarize your work in Parts a–c in an if-then statement that begins as follows:

In a triangle, if the lengths of two sides and the measure of the angle included between those sides are known, then … .

2 Next, examine $\triangle ABC$ and $\triangle XYZ$ shown below. In both cases, you have information given about two sides and an included angle. In $\triangle XYZ$, k is a constant. Test if this information is sufficient to conclude that $\triangle ABC \sim \triangle XYZ$ using Parts a–d as a guide.



a. In $\triangle XYZ$, $m\angle X = m\angle A = 60°$, $z = k \cdot c = 12k$ cm, and $y = k \cdot b = 18k$ cm, where k is a constant. Explain why x, $m\angle Y$, and $m\angle Z$ are uniquely determined.

b. Use the information given for $\triangle XYZ$ to write and then solve an equation to find length x. Based on your work in Problem 1, how is x related to a?

c. Find m∠Y. How are m∠B and m∠Y related?

d. Find m∠Z. How are m∠C and m∠Z related?

e. Considering all the information in Parts a–d, what can you conclude about $\triangle ABC$ and $\triangle XYZ$?

3 Suppose you know that in $\triangle ABC$ and $\triangle DEF$, m∠C = m∠F, $d = k \cdot a$, and $e = k \cdot b$.

a. Could you prove that $\triangle ABC \sim \triangle DEF$? If so, explain how. If not, explain why not.

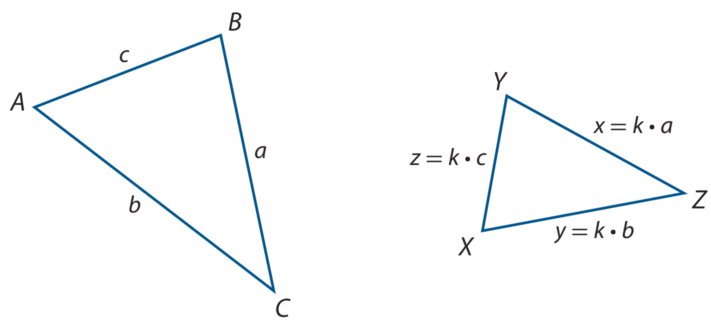
b. Complete the following statement.

If an angle of one triangle has the same measure as an angle of a second triangle, and if the lengths of the corresponding sides including these angles are multiplied by the same scale factor k, then … .

c. Compare your statement in Part b with those of your classmates. Resolve any differences.

The conclusion you reached in Part c of Problem 3 is called the **Side-Angle-Side (SAS) Similarity Theorem**. This theorem gives at least a partial answer to the question of finding minimal conditions that will guarantee two triangles are similar. In the next several problems, you will explore other sets of sufficient conditions.

4 Suppose you know that in $\triangle ABC$ and $\triangle XYZ$, the lengths of corresponding sides are related by a scale factor k as in the diagram below.



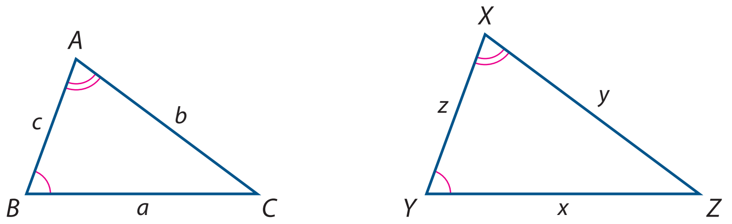
a. What additional relationship would you need to know in order to conclude that $\triangle ABC \sim \triangle XYZ$? What is a possible strategy you could use to establish the relationship?

b. Write a deductive argument proving the relationship you stated in Part a. What can you conclude?

c. The results of your work in Parts a and b establish a **Side-Side-Side (SSS) Similarity Theorem**. Write this SSS Similarity Theorem in if-then form.

5 In Problems 2 and 4, given two triangles, you needed to know three of their corresponding measures in order to determine that the triangles were similar. In this problem, you will examine a situation in which only two corresponding measures of two triangles are known.

Suppose you are given $\triangle ABC$ and $\triangle XYZ$ in which $m\angle X = m\angle A$ and $m\angle Y = m\angle B$.



a. Do you think $\triangle ABC \sim \triangle XYZ$? Why or why not?

b. What additional relationship(s) would you need to know to conclude for sure that the triangles are similar?

c. Students at Madison East High School believed the answer to Part a was "yes." Study their proof. Provide reasons that support each of their statements.

Proof:

We first looked at the ratio of the lengths a and x of a pair of corresponding sides.

If we let k represent the ratio $\frac{x}{a}$, then $x = ka$. (1)

For $\triangle ABC$, we know that $\frac{a}{\sin A} = \frac{b}{\sin B}$. So, $b = \frac{a \sin B}{\sin A}$. (2)

Similarly for $\triangle XYZ$, we know that $\frac{x}{\sin X} = \frac{y}{\sin Y}$. So, $y = \frac{x \sin Y}{\sin X}$. (3)

It follows that $y = \frac{ka \sin Y}{\sin X}$. (4)

Since $m\angle Y = m\angle B$ and $m\angle X = m\angle A$, it follows that $y = \frac{ka \sin B}{\sin A}$. (5)

Since $b = \frac{a \sin B}{\sin A}$, it follows that $y = kb$. (6)

Since $m\angle X = m\angle A$ and $m\angle Y = m\angle B$, then $m\angle Z = m\angle C$. (7)

We've shown $x = ka$, $y = kb$, and $m\angle Z = m\angle C$.

So, $\triangle ABC \sim \triangle XYZ$. (8)

d. Could the students have reasoned differently at the end of their argument? How?

e. Write an if-then statement of the theorem proved in this problem. What name would you give the theorem?

Summarize the Mathematics

In this investigation, you established three sets of conditions, each of which is sufficient to prove that two triangles are similar.

a State, and illustrate with diagrams, the sets of conditions on side lengths and/or angle measures of a pair of triangles that ensure the triangles are similar.

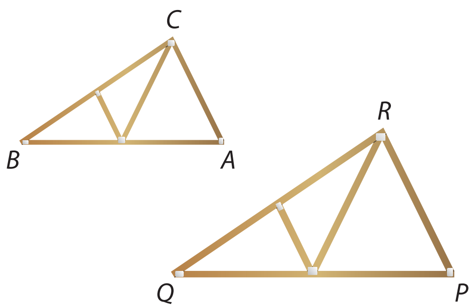
b For each set of conditions, identify the key assumptions and theorems used in deducing the result.

c How did the Law of Cosines and Law of Sines help in determining the sets of sufficient conditions?

Be prepared to share your descriptions and reasoning with the class.

Check Your Understanding

The design of homes and office buildings often requires roof trusses that are of different sizes but have the same pitch. The trusses shown below are designed for roofs with solar hot-water collectors. For each Part a through e, suppose a pair of roof trusses have the given characteristics. Determine if $\triangle ABC \sim \triangle PQR$. If so, explain how you know that the triangles are similar and give the scale factor from $\triangle ABC$ to $\triangle PQR$. If not, give a reason for your conclusion.



a. $m\angle A = 57°$, $m\angle B = 38°$, $m\angle P = 57°$, $m\angle R = 95°$

b. $AB = 12$, $BC = 15$, $m\angle B = 35°$, $PQ = 16$, $QR = 20$, $m\angle Q = 35°$

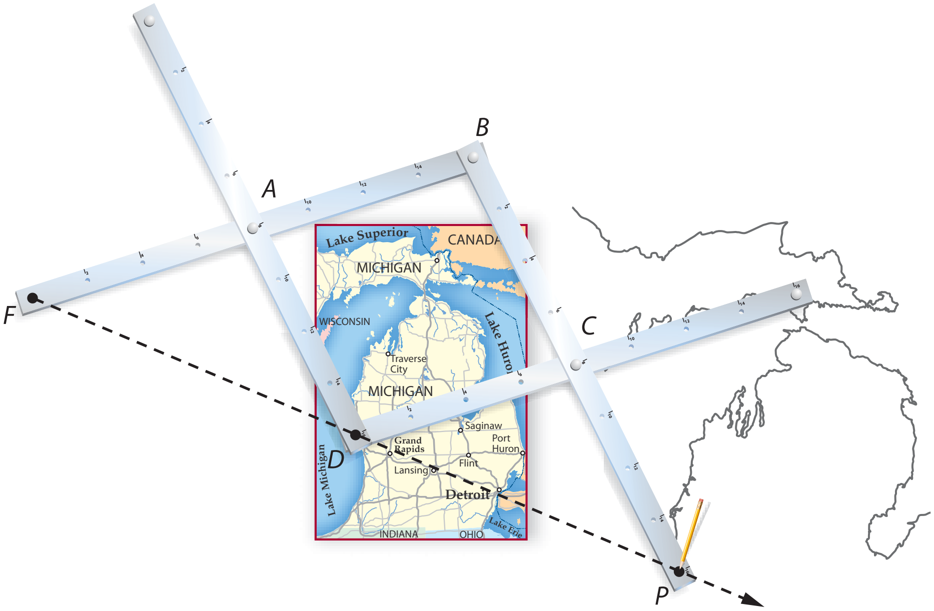
c. $AB = BC$, $m\angle B = m\angle Q$, $PQ = QR$

d. $AC = 4$, $BC = 16$, $BA = 18$, $PR = 10$, $QR = 40$, $QP = 48$

e. $AB = 12$, $m\angle A = 63°$, $m\angle C = 83°$, $m\angle P = 63°$, $m\angle Q = 34°$, $PQ = 12$

Investigation 3 Reasoning with Similarity Conditions

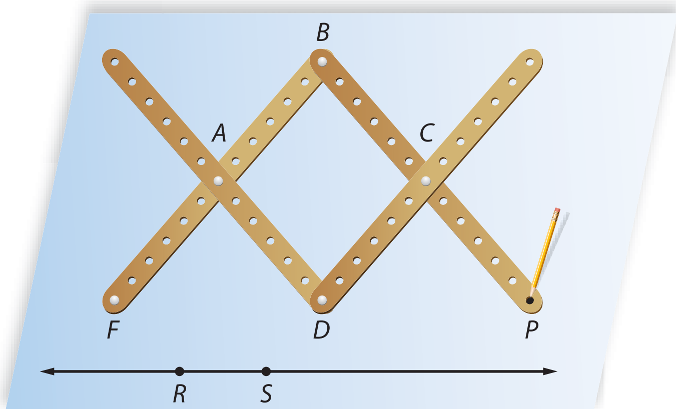
Similarity and proportionality are important ideas in mathematics. They provide tools for designing mechanisms, calculating heights and distances, and proving mathematical relationships. For example, a *pantograph* is a parallelogram linkage used for copying drawings and maps to a different scale. The pantograph is fastened down with a pivot at F. A stylus is inserted at D. A pencil is inserted at P. As the stylus at D traces out a path on the map, the pencil at P draws a scaled copy of that path.



As you work on the problems of this investigation, look for answers to why a pantograph works as it does and the more general question:

What strategies are useful in solving problems using similar triangles?

1 **Designing Pantographs** Working with a partner, construct a model of a pantograph like the one shown below. Connect four linkage strips at points A, B, C, and D with paper fasteners.



a. Mark points R and S on a sheet of paper about two inches apart. Draw $\overleftrightarrow{RS}$. Place the pantograph so that point F is on the line near the left end. Label your placement of point F for future reference. Hold point F *fixed*. Place point D somewhere else on the line. Where is point P in relation to the line?

As you move point D along the line:

i. describe the path a pencil at point P follows.

ii. what kind of quadrilateral is $ABCD$? Why?

iii. how is the triangle determined by points F, B, and P related to the triangle determined by points F, A, and D? Explain your reasoning.

iv. what can you conclude about the lengths FP and FD?

b. Now, holding point F fixed in the same place as before, find the image of point S by placing point D over point S and marking the location of point P. Label your mark as the image point S'. What is true about points F, S, and S'?

c. Keeping F fixed in the same position, find the image of point R in the same manner. Label it R'.

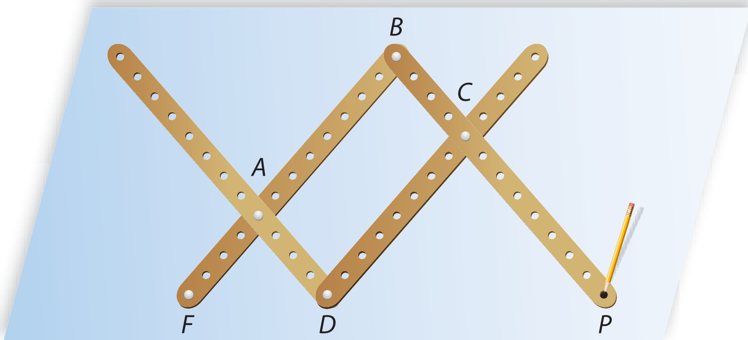
i. What is true about the points F, R, and R'?

ii. Compare the lengths RS and $R'S'$. How does the relationship compare with what you established in Part aiii?

d. Mark a point T on your paper so that $\triangle RST$ is a right triangle with right angle at R. Find the image of point T as you did for points R and S.

e. How is $\triangle R'S'T'$ related to $\triangle RST$? Explain your reasoning.

2 Examine the pantograph illustrated in the diagram below.



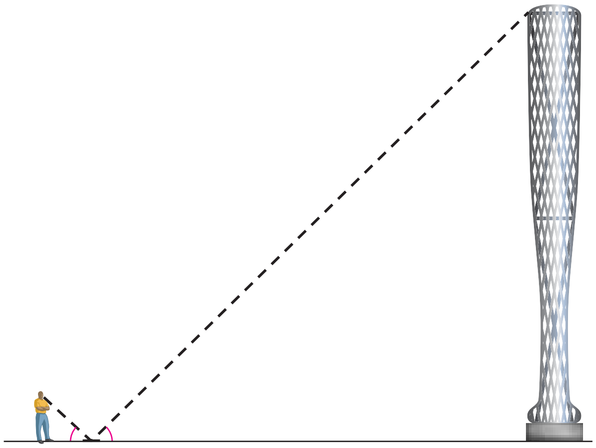
a. Draw a segment, $\overline{XY}$, that is 5 cm long. Imagine drawing the image of $\overline{XY}$ with this pantograph. Predict the length of the image $\overline{X'Y'}$. Explain the basis for your prediction and then check it using your pantograph.

b. How would you assemble your pantograph so that it multiplies distances by 4?

c. How will the pantograph affect distances if it is assembled so that $FB = k \cdot FA$? Why?

3 **Calculating Heights and Distances** In Course 2, you calculated the height of Chicago's *Bat Column* and other structures using trigonometry. Triangle similarity provides another method.

Suppose a mirror is placed on the ground as shown. You position yourself to see the top of the sculpture reflected in the mirror. An important property of physics states that in such a case, the *angle of incidence* is congruent to the *angle of reflection*.



a. On a copy of the diagram above, sketch and label a pair of triangles which, if proven similar, could be used to calculate the height of the *Bat Column*. Write a similarity statement relating the triangles you identified.

b. Prove that the two identified triangles are similar.

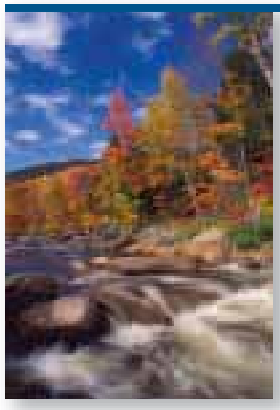
c. Add to your diagram the following measurements.

- The ground distance between you and the mirror image of the top of the column is 6 feet 5 inches.
- The ground distance between the mirror image and the base of the column is 116 feet 8 inches.
- Assume that the distance from the ground up to your eyes is 66 inches.

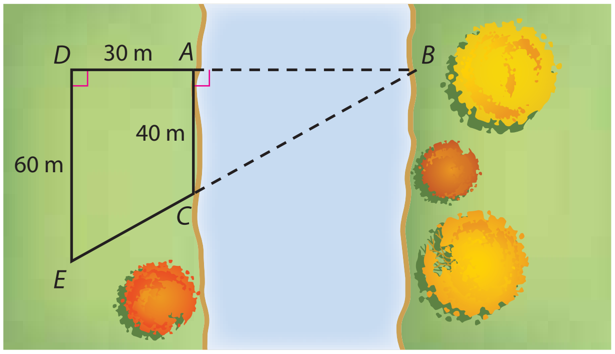
About how tall is the column?

d. How could you use a trigonometric ratio and the ground distance from the mirror image to the column to calculate the height of the *Bat Column*? What measurement would you need? How could you obtain it?

4 As part of their annual October outing to study the changing colors of trees in northern Maine, several science club members from Poland Regional High School decided to test what they were learning in their math class by finding the width of the Penobscot River at a particular point *A* as shown on the next page.



Pacing from point A, they located points D, E, and C as shown in the diagram below.



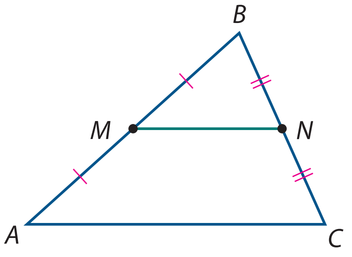
a. How do you think they used similarity to calculate the distance AB? Be as precise as possible in your answer.

b. What is your estimate of the width of the river at point A?

c. Another group of students repeated the measurement, again pacing from point A to locate points D, E, and C. In this case, $AD = 18$ m, $DE = 26$ m, and $AC = 20$ m. Would you expect that this second group got the same estimate for the width of the river as you did in Part b? Explain.

d. What trigonometric ratio could be used to calculate the width of the river? What measurements would you need and how could you obtain them?

5 **Discovering and Proving New Relationships** Study the diagram below of $\triangle ABC$. $\overline{MN}$ connects the midpoints M and N of sides $\overline{AB}$ and $\overline{BC}$, respectively.



a. How does $\overline{MN}$ appear to be related to $\overline{AC}$?

b. How does the length of $\overline{MN}$ appear to be related to the length of $\overline{AC}$?

c. Use interactive geometry software or careful paper-and-pencil drawings to investigate if your observations in Parts a and b hold for triangles of different shapes. Compare your findings with those of your classmates.

d. Complete the following statement.

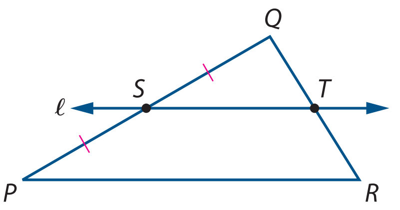
If a line segment joins the midpoints of two sides of a triangle, then it … .

e. Collaborating with others as needed, write a proof of the statement in Part d. That statement is often called the **Midpoint Connector Theorem for Triangles**.

6 Possible new theorems are often discovered by studying several cases and looking for patterns as you did in Problem 5. Another way of generating possible theorems is to modify the statement of a theorem you have already proven. For example, consider the following statement.

If a line intersects the midpoint of one side of a triangle and is parallel to a second side, then it intersects the third side at its midpoint.

Do you think this statement is always true? If so, prove it. If not, provide a counterexample.

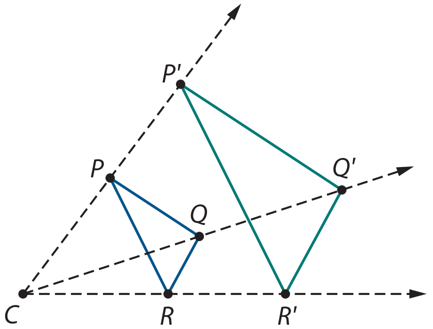


Proving Properties of Size Transformations

In Course 2, you studied size transformations with center at the origin, defined by coordinate rules of the form $(x, y) \rightarrow (kx, ky)$, where $k > 0$. Your work with pantographs suggests a way of defining size transformations without using coordinates and with *any* point in the plane as the center.

For any point C and real number $k > 0$, a **size transformation** with **center** C and **magnitude** k is a function that maps each point of the plane onto an image point as follows:

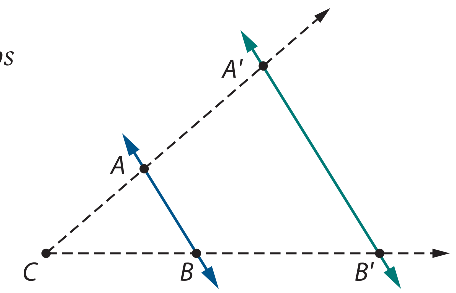
- Point C is its own image.
- For any point P other than C, the image of point P is the point P' on $\overrightarrow{CP}$ for which $CP' = k \cdot CP$.



7 Use the definition of a size transformation and the diagram below to prove each statement.

If a size transformation with center C and magnitude k maps A onto A′ and B onto B′, then:

- $A'B' = k \cdot AB$
- $\overleftrightarrow{A'B'} \parallel \overline{AB}$

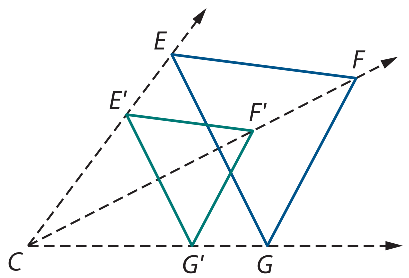


8 In the diagram below, $\triangle E'F'G'$ is the image of $\triangle EFG$ under a size transformation with center C and magnitude k.

a. What is true of magnitude k?

b. How do $\angle FEG$ and $\angle F'E'G'$ appear to be related?

c. How do $\triangle EFG$ and $\triangle E'F'G'$ appear to be related?



d. Use the diagram (bottom of page 177) and the results of Problem 7 to help you prove these generalizations of your observations.

- *Under a size transformation, a triangle and its image are similar.*
- *Under a size transformation, an angle and its image are congruent.*

Summarize the Mathematics

In this investigation, you explored how conditions for similarity of triangles could be used to solve applied problems and to prove important mathematical relationships.

a How would you assemble a pantograph to make an enlargement of a shape using a scale factor of 8? Of r?

b How could you calculate the height of a structure like the *Bat Column* using shadows cast on a sunny day?

c Suppose in a diagram, $\triangle M'N'O'$ is the image of $\triangle MNO$ under a size transformation of magnitude $k \neq 1$.

  i. How could you locate the center of the size transformation?

  ii. How could you determine the magnitude k?

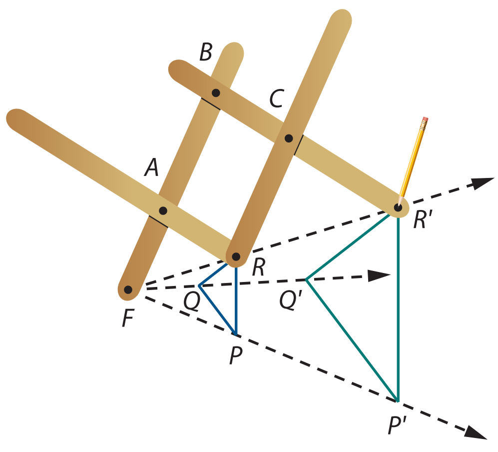
  iii. How are the side lengths and angle measures of the two triangles related?

d What strategies are helpful in using similarity to solve problems in applied contexts? To prove mathematical statements?

Be prepared to explain your ideas and strategies to the class.

Check Your Understanding

Examine the diagram of a pantograph enlargement of $\triangle PQR$.



a. Prove each statement.

  i. $\triangle FAR \sim \triangle FBR'$

  ii. $\triangle FRQ \sim \triangle FR'Q'$

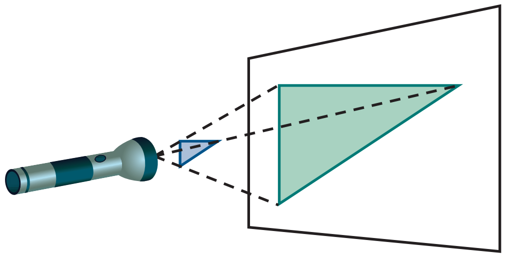
  iii. $\triangle PQR \sim \triangle P'Q'R'$

b. How could you represent the scale factor of this enlargement?

c. Describe the center and magnitude of a size transformation that will map $\triangle P'Q'R'$ onto $\triangle PQR$.

Applications

1 A flashlight is directed perpendicularly at a vertical wall 24 cm away. A cardboard triangle with sides of lengths 3, 4, and 5 cm is positioned directly between the light and the wall, parallel to the wall so that its projected shadow image is similar to it.



a. Suppose the shadow of the 4-cm side is 10 cm. Find the lengths of the shadows of the other two sides.

b. Suppose the shadow of the 5-cm side is 7.5 cm. Find the lengths of the other two sides of the shadow.

c. How far from the light source should you place the cardboard triangle so that the 4-cm side has a 12-cm shadow?

2 While pumping gas on his way to work, Josh witnessed a robber run out of the station, get into his car, and drive off. An off-duty police detective, Josh knew that he needed to document the scene before the evidence was destroyed. Using his cell phone camera and whatever he could find in his pockets, Josh took the following four pictures.

a. Consider the dimensions of the standard dollar bill in each of the photos. In each case below, which photo provides more useful or reliable information? Explain your reasoning.

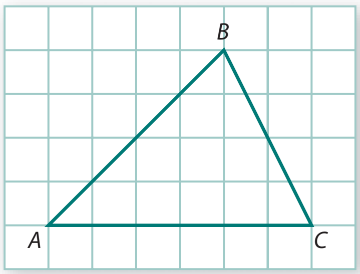
i. Photo 1 or Photo 2

ii. Photo 3 or Photo 4

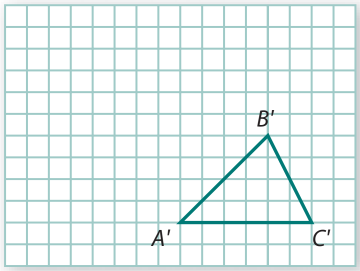
b. How could a technician use your selected photos to calculate the actual dimensions of the impression? What are those dimensions?

c. Would you need to consider two photos to determine the actual dimensions of the impression, or could you use just one? How might having two different photos strengthen the case?

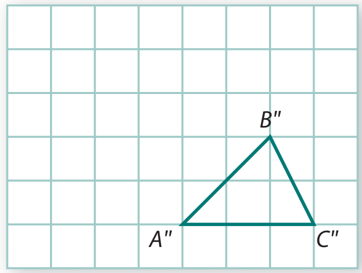
3 Amber and Christina had different ideas about how to use grid paper to reduce triangles to half their original size. Each was given $\triangle ABC$ drawn on half-inch grid paper. Amber drew $\triangle A'B'C'$ as shown on quarter-inch grid paper. Christina drew $\triangle A''B''C''$ as shown on half-inch grid paper.



Original Triangle



Amber's Triangle



Christina's Triangle

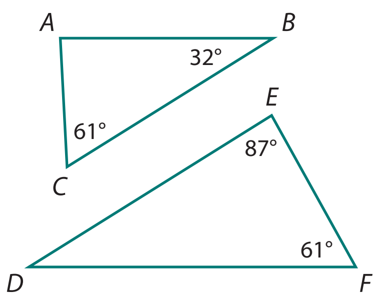
a. Explain why Amber's triangle is similar to the original. What is the scale factor?

b. Explain why Christina's triangle is similar to the original and has a scale factor of $\frac{1}{2}$.

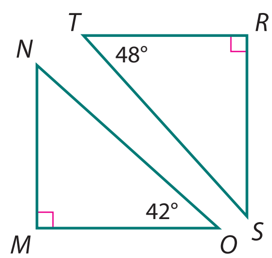
c. Are Christina's triangle and Amber's triangle similar to one another? If so, what is the scale factor?

4 Examine each pair of triangles. Using the information given, determine whether the triangles are similar or the information is inconclusive. Explain your reasoning.

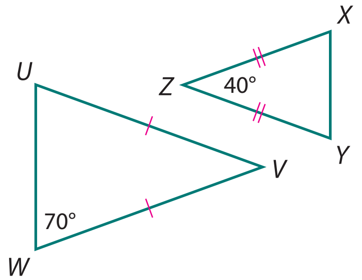
a.



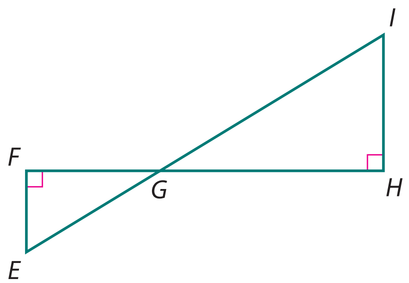
b.



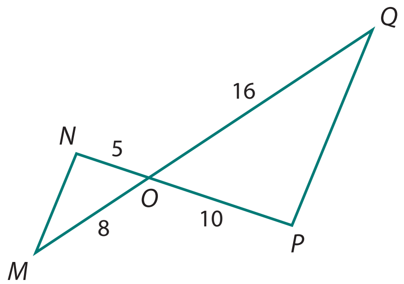
c.



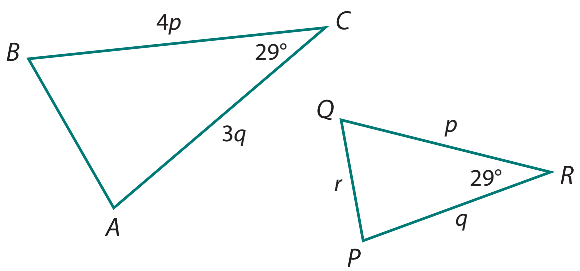
d.



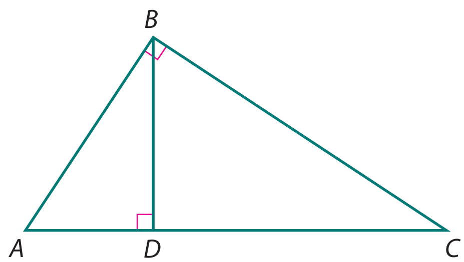
e.



f.

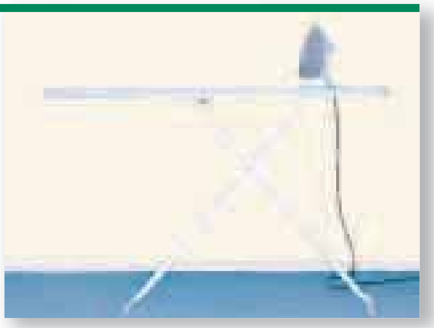


5 In the diagram below, $\triangle ABC$ is a right triangle. $\overline{BD}$ is the altitude drawn to the hypotenuse $\overline{AC}$.



a. Identify three pairs of similar triangles. Be sure that corresponding vertices are labeled in the same order.

b. Describe the strategies you would use to prove each pair of triangles similar.



6 Most ironing boards can be adjusted to different heights. One ironing board, with a design similar to the one shown here, has legs that are each 110 cm long and are hinged at a point that is 40 cm from the top end of each. Possible working heights are 90 cm, 85 cm, 80 cm, 75 cm, and 70 cm.

a. Make a sketch of the ironing board, labeling vertices of important triangles. Carefully explain why, for any of the working heights, the surface of the ironing board is parallel to the floor.

b. For a working height of 90 cm, determine the distance between the two points at which the legs connect to the ironing board. Find the measures of the angles of the top triangle.

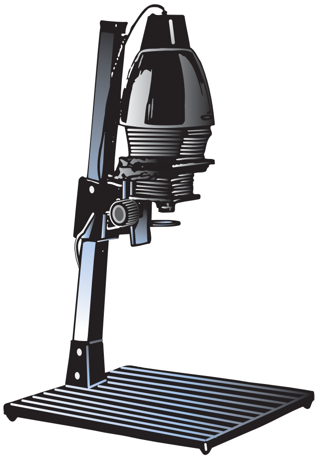
7 In Investigation 2, you established three sets of conditions that are sufficient to prove that two triangles are similar: SAS, SSS, and AA. Here, you are asked to replace the word "triangle" in these statements with the word "parallelogram." Then decide if the new statement is true or not.

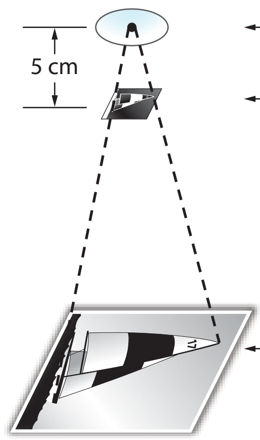
a. Rewrite the SAS Similarity Theorem for triangles as a statement about two parallelograms. Prove or disprove your statement.

b. Rewrite the SSS Similarity Theorem for triangles as a statement about two parallelograms. Prove or disprove your statement.

c. Rewrite the AA Similarity Theorem for triangles as a statement about two parallelograms. Prove or disprove your statement.

8 Wildlife photographers use sophisticated film cameras in their field work. Negatives from the film are then resized for publication in books and magazines. In this task, you will examine how a photographic enlarger uses the principles of size transformations. The light source is the center of the size transformation, a photographic negative or slide is the preimage, and the projection onto the photographic paper is the image. On one enlarger, the distance between the light source and film negative is fixed at 5 cm. The distance between the paper and the negative adjusts to distances between 5 cm and 30 cm.





a. If the distance from the light source to the paper is 20 cm, what ratio gives the magnitude of the size transformation represented?

b. If the distance between the negative and the paper is 8 cm, what is the magnitude of the size transformation?

c. Deonna uses 35 mm film in her camera, so her negatives measure 35 mm by 23 mm.

i. When the paper is 5 cm from the negative, her prints are 70 mm by 46 mm (7.0 cm by 4.6 cm). Use a diagram and your understanding of size transformations to explain why this is the case.

ii. When she adjusts the distance between the paper and the negative to 10 cm, her prints are 105 mm by 69 mm (10.5 cm by 6.9 cm). Explain why this is the case.

iii. What is the magnitude of the enlargement in each of the two settings above? How does this affect the distances involving the light source, negative, and photographic paper?

d. Recall that the distance between the paper and the negative can be adjusted to distances from 5 cm to 30 cm. What are the dimensions of the *smallest* print that Deonna can make from a negative using this brand of enlarger? What are the dimensions of the *largest* print?

e. Suppose the paper is 20 cm from Deonna's negative. What size print is produced?

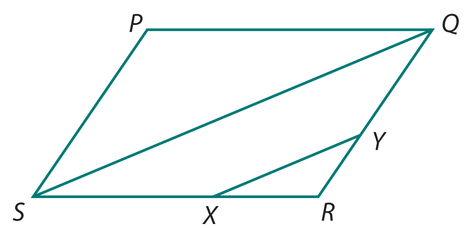
f. How far should the paper be from Deonna's negative to produce a print whose longest side is 12 cm?

9 In the diagram below, quadrilateral $PQRS$ is a parallelogram, $\overline{SQ}$ is a diagonal, and $\overleftrightarrow{SQ} \parallel \overleftrightarrow{XY}$.

a. Prove that $\triangle XYR \sim \triangle SQR$.

b. Prove that $\triangle XYR \sim \triangle QSP$.

c. Identify the center and magnitude of a size transformation that maps $\triangle RXY$ onto $\triangle RSQ$.



10 Using interactive geometry software (without coordinates) or paper and pencil, conduct the following experiment.

- Draw $\triangle ABC$.
- Mark a point D and then find the image of $\triangle ABC$ under a size transformation with center at D and scale factor 3. Label the image $\triangle A'B'C'$.
- Mark a second point E and then find the image of $\triangle A'B'C'$ under a size transformation with center E and scale factor $\frac{1}{2}$. Label the image $\triangle A''B''C''$.

a. How are $\triangle ABC$ and $\triangle A''B''C''$ related? Explain your reasoning.

b. Can you find a single size transformation that maps $\triangle ABC$ onto $\triangle A''B''C''$? If so, what is the center of the size transformation? What is the scale factor?

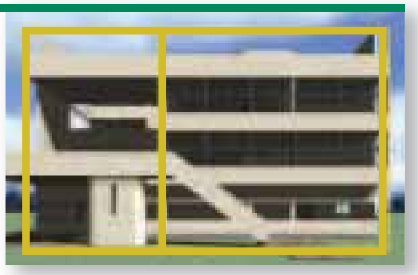
c. Repeat the experiment for a different $\triangle ABC$, different points D and E, and different scale factors. Include a case where the two centers are the same point. Answer Parts a and b for these new cases.

d. Repeat Part c. But this time, use scale factors 2 and $\frac{1}{2}$.

e. Conduct additional experiments as necessary in search of general patterns to complete this statement.

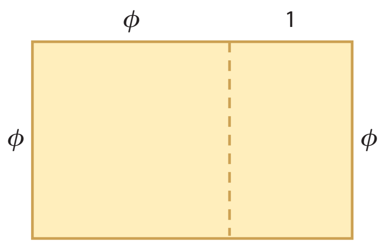
The composition of two size transformations with scale factors m and n is … .

Connections



The French architect Le Corbusier incorporated the golden rectangle in the design of this villa.

11 **Golden rectangles** have the special property that if you cut off a square from one side, the remaining rectangle is similar to the original rectangle. The ratio $\frac{\text{length of long side}}{\text{length of short side}}$ leads to the special proportion,



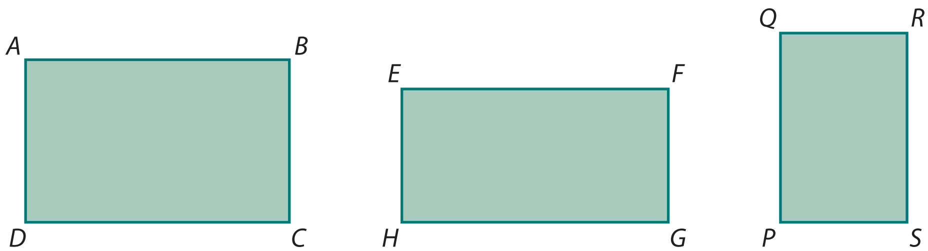
$$\frac{\phi}{1} = \frac{\phi + 1}{\phi}, \text{ or}$$

$$\phi = \frac{\phi + 1}{\phi}.$$

The number ϕ (the Greek letter "phi") is called the **golden ratio**. Golden rectangles are often judged to be the most attractive rectangles from an artistic and architectural point of view.

a. Use algebraic reasoning to determine exact and approximate values of ϕ.

b. Determine which of the rectangles below are golden rectangles. Describe your test.



c. Is rectangle *ABCD* similar to rectangle *PQRS*? Explain your reasoning.

d. Prove or disprove that all golden rectangles are similar.

12 Suppose $\triangle ABC \sim \triangle XYZ$ with k as the scale factor from $\triangle ABC$ to $\triangle XYZ$.

a. Explain why $\frac{XY}{AB} = \frac{YZ}{BC} = \frac{XZ}{AC}$.

b. Explain why $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$.

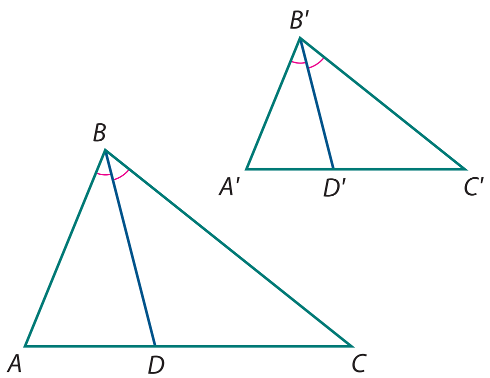
c. Recall that a *proportion* is a statement of equality between ratios. Restate the definition of similar polygons (page 165) using the idea of proportion.

13 In similar triangles, the ratios of lengths of the corresponding sides are equal.

CPMP-Tools

a. Use interactive geometry software to test each of the following claims by a group of students at Clayton High School. Which claims appear to be true statements?

Claim 1: In similar triangles, the ratio of the lengths of the bisectors of corresponding angles is the same as the ratio of the lengths of corresponding sides. For example, if $\triangle ABC \sim \triangle A'B'C'$, $\overline{BD}$ bisects $\angle ABC$, and $\overline{B'D'}$ bisects $\angle A'B'C'$, then $\frac{BD}{B'D'} = \frac{AC}{A'C'}$.



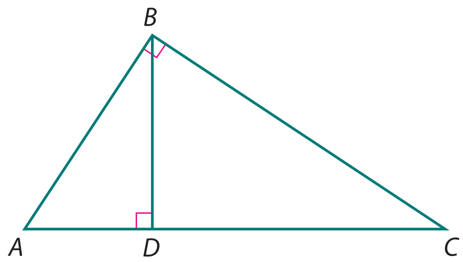
Claim 2: In similar triangles, the ratio of the lengths of the altitudes from corresponding vertices is the same as the ratio of the lengths of corresponding sides.

Claim 3: In similar triangles, the ratio of the lengths of the medians from corresponding vertices is the same as the ratio of the lengths of corresponding sides.

b. Write an argument proving one of the claims that you think is true.

c. Restate the claim you proved in Part b using the language of proportions.

14 Look back at your work for Applications Task 5. Since $\triangle ABD \sim \triangle BCD$, it follows that $\frac{AD}{BD} = \frac{BD}{CD}$. Note that BD appears twice in the proportion. The length of $\overline{BD}$ is the *geometric mean* of the lengths of $\overline{AD}$ and $\overline{CD}$.



a. Using the language of geometric mean, state a theorem about the altitude to the hypotenuse of any right triangle.

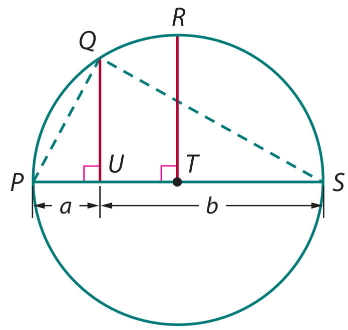
b. The **geometric mean** of two positive numbers a and b is the positive number x such that $\frac{a}{x} = \frac{x}{b}$, or $x = \sqrt{ab}$. What is the geometric mean of 4 and 9? Of 7 and 12?

c. For any two positive numbers, describe the relation between the arithmetic mean and the geometric mean using $<$, $\leq$, $=$, $>$, or $\geq$.

d. At the right is a circle with center T.

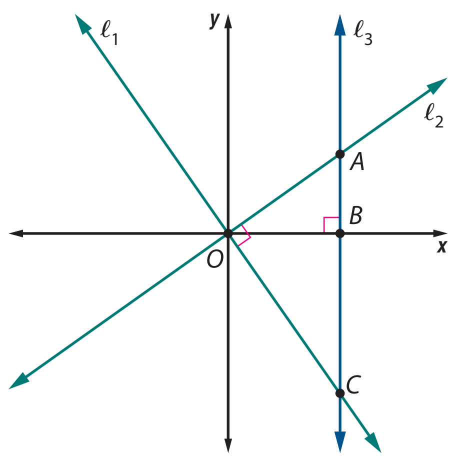
i. Explain why one of the segments, $\overline{QU}$ and $\overline{RT}$, has length the geometric mean of a and b and the other has length the arithmetic mean of a and b.

ii. How could you use the diagram to justify your answer to Part c?



e. Under what circumstances are the arithmetic mean and geometric mean of a and b equal?

15 In the Course 2 *Coordinate Methods* unit, you discovered that two nonvertical lines in a coordinate plane are perpendicular if and only if their slopes are opposite reciprocals. In this task, you will prove that *if* the lines are perpendicular, *then* the slopes are opposite reciprocals. In Extensions Task 27, you will prove the converse. You now have the necessary tools to prove these claims. In the diagram at the right, $\ell_1 \perp \ell_2$.



a. Why can line ℓ_3 be drawn through point A on ℓ_2 perpendicular to the x-axis?

b. What is the slope of ℓ_1? Of ℓ_2?

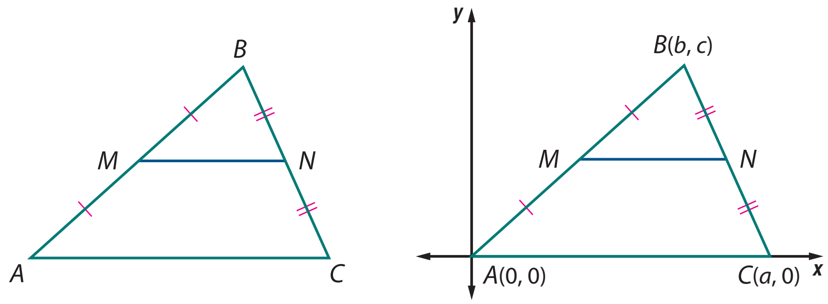
c. Prove that $\triangle AOB \sim \triangle OCB$.

d. Using Parts b and c, justify that the slopes of ℓ_1 and ℓ_2 are opposite reciprocals.

16 In Problem 5 of Investigation 3, you used similar triangles to prove this statement.

If a line segment joins the midpoints of two sides of a triangle, then it is parallel to the third side and one half its length.

Your proof involved reasoning from a combination of assumed or established statements that were independent of a coordinate system. Such proofs are sometimes called *synthetic proofs*.



a. Use the labeled diagram on the right above to help you write a proof, using coordinates, that $\overleftrightarrow{MN} \parallel \overleftrightarrow{AC}$.

b. Use coordinate methods to prove $MN = \frac{1}{2}AC$.

c. Compare the proofs you constructed in Parts a and b with those you constructed in Problem 5 (page 176).

- Which proof was easier for you to construct? Why?
- Which proof would be easier for you to explain to someone else? Why?

d. Suppose M is $\frac{1}{3}$ the distance from point B to point A and N is $\frac{1}{3}$ the distance from point B to point C. How is $\overline{MN}$ related to $\overline{AC}$?

e. Write a proof of your conjecture in Part d. Did you use synthetic or coordinate methods? Why?

17 There are important connections between the perimeters of similar figures and between the areas of similar figures.

a. Suppose $\triangle P'Q'R'$ is the image of $\triangle PQR$ under a size transformation with center C and magnitude k.

   i. Write an argument to prove that *perimeter* $\triangle P'Q'R' = k(\textit{perimeter } \triangle PQR)$.

   ii. Use a result from Connections Task 13 to help prove that *area* $\triangle P'Q'R' = k^2(\textit{area } \triangle PQR)$.

b. Suppose one polygon is the image of another polygon under a size transformation with center C and magnitude k.

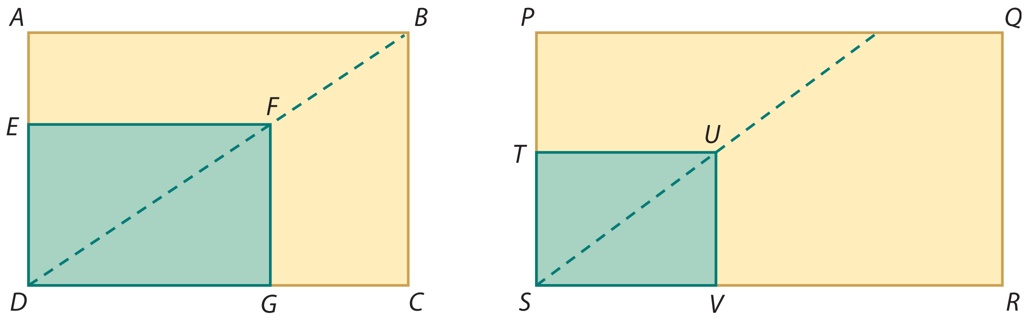
   i. How are their perimeters related? Explain your reasoning.

   ii. How are their areas related? Explain your reasoning.

On Your Own

Reflections

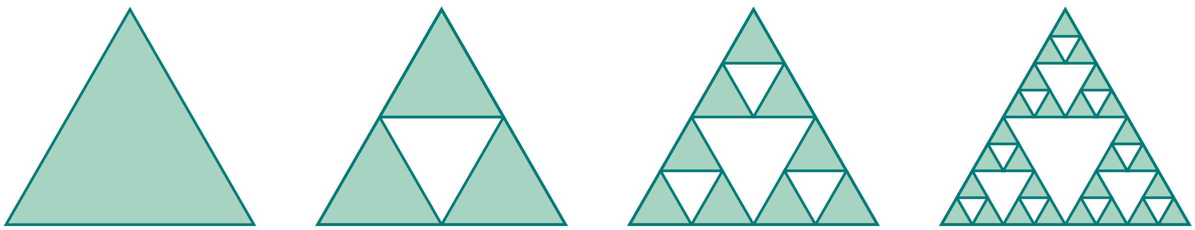
18 A quick test that engravers and photographers use to determine whether two rectangular shapes are similar is illustrated in the diagram below.



a. Explain why *rectangle ABCD* ~ *rectangle EFGD* but rectangle *PQRS* is not similar to rectangle *TUVS*.

b. Can this *diagonal test* be used to determine if two nonrectangular parallelograms are similar? Explain your reasoning.

19 A *Sierpinski triangle* is constructed through a sequence of steps illustrated by the figures below. At Stage $n = 0$, you construct an equilateral triangle whose sides are all of length 1 unit. In succeeding stages, you remove the "middle triangles," from each of the remaining colored triangles as shown in Stages $n = 1$, 2, and 3. This process continues indefinitely.



The Sierpinski triangle is an example of a *fractal* in that small portions of the design are similar to the design as a whole.

a. How is the idea of "self-similarity" seen at Stage 3?

b. Using the Stage 3 triangle, attach, or imagine attaching, a congruent copy of that triangle to each side of the Stage 3 triangle. How is the larger triangle formed related to the stages of construction?

c. In what sense can the triangle at Stage 0 be considered similar to itself?

d. How many colored similar triangles are formed at Stage 1? At Stage 2?

e. Use a computer program like the interactive geometry "Sierpinski Triangle" sketch to explore cases where the initial triangle is not an equilateral triangle. How, if at all, would you change your answers for Parts a–d?

20 Look back at the pantograph setup in Investigation 3, Problem 1 (page 173). Suppose the tracing stylus at point D and the pencil at point P were interchanged. Suppose S' is the image produced when tracing a shape S with the stylus so positioned.

a. How are shapes S and S' related?

b. How would you modify your answer in the case of the pantograph shown in Problem 2 (page 174)?

21 In Problem 5 of Investigation 3, you wrote a synthetic proof of the Midpoint Connector Theorem for Triangles. In Connections Task 16, you prepared a coordinate proof of the theorem.

a. Using the diagram in Problem 5 (page 176), describe how the theorem could be proved using a size transformation with center at vertex B of $\triangle ABC$.

b. If you were asked to demonstrate a proof of this theorem later in the year, which proof method would you use? Why?

22 How could you use a size transformation to explain why a circle of radius 5 is similar to a circle of radius 12? Why any two circles are similar?

Extensions

23 Pietro's Pizza makes rectangular pizzas. The shop posted an inviting message on its sign, "Now! Our large is 20% bigger!" The original large pizza had dimensions 20 inches by 15 inches.

a. Assume 20% bigger means that the dimensions of the new pizza are 120% of the original size.

i. Why is the new pizza similar (in shape) to the original pizza?

ii. What is the scale factor relating the original pizza to the new pizza?

iii. What are the dimensions of the new pizza?

iv. Calculate the ratio of the perimeter of the new pizza to that of the original. How is this ratio related to the scale factor?

v. Calculate the ratio of the areas of the new pizza to the original pizza. How is this ratio related to the scale factor?

b. Assume 20% bigger means that the area of the new pizza is 120% of the original size.

i. What is the area of the new pizza?

ii. If the new pizza is similar to the original, what are its dimensions?

iii. What is the scale factor relating the original pizza to the new pizza?

c. Which interpretation favors the consumer?

24 Study the following algorithm for constructing a golden rectangle. (See Connections Task 11.)

Step 1. Construct square $ABCD$.

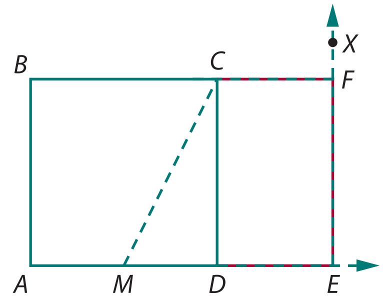
Step 2. Locate the midpoint M of $\overline{AD}$.

Step 3. Draw $\overline{CM}$.

Step 4. Locate point E on $\overrightarrow{AD}$ so that $MC = ME$.

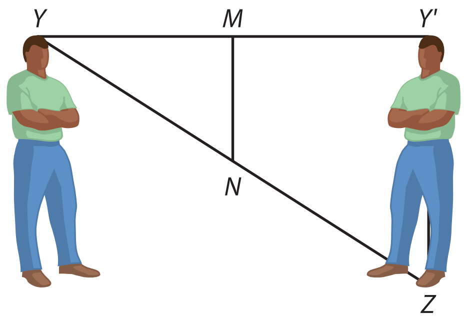
Step 5. Construct $\overrightarrow{EX} \perp \overline{AE}$.

Step 6. Extend $\overline{BC}$ to intersect $\overrightarrow{EX}$ at point F.



Prove that rectangle $ABFE$ is a golden rectangle.

25 Using your knowledge of mirror reflections and the diagram below (in which $\overline{MN}$ represents the mirror), determine the wall mirror of minimum height that would permit you to see a full view of yourself. State any assumptions you made.



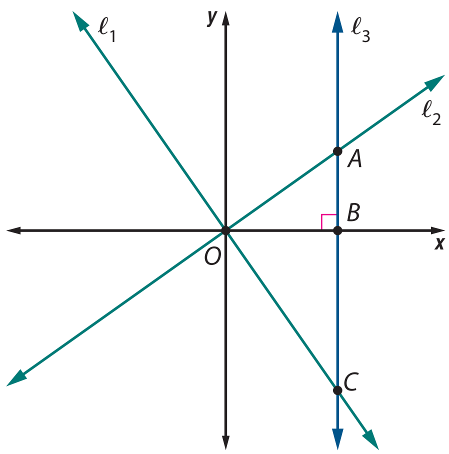
26 In Connections Task 14, you conjectured and then provided a geometric proof that if a and b are two positive numbers, their arithmetic mean is greater than or equal to their geometric mean. Use the facts that $(a + b)^2 \geq 0$ and $(a - b)^2 \geq 0$ and algebraic reasoning to prove the **arithmetic-geometric mean inequality**, $\frac{a + b}{2} \geq \sqrt{ab}$.

27 In Connections Task 15, you proved that if two nonvertical lines are perpendicular, then their slopes are opposite reciprocals.

Using the diagram at the right, prove this converse statement.

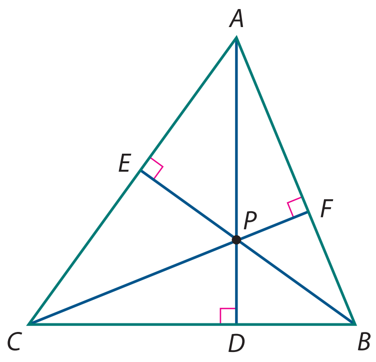
If the slopes of two nonvertical lines are negative reciprocals, then the lines are perpendicular.

Assume the slope of ℓ_1 is m_1 and the slope of ℓ_2 is m_2, where $m_1 m_2 = -1$.



28 Using interactive geometry software, draw $\triangle ABC$ with altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.

a. Calculate the product $\left(\frac{AF}{FB}\right)\left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)$.

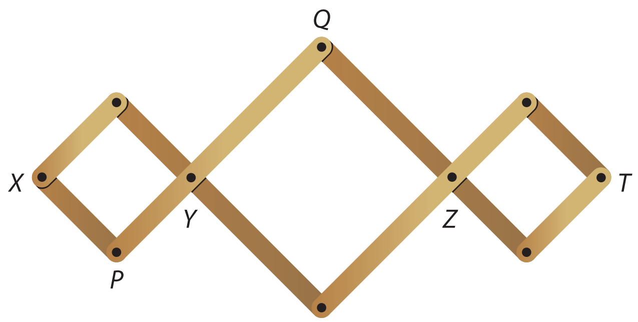


b. Use the dragging capability of your software to explore if the relationship you found in Part a holds for other triangles.

c. Make a conjecture based on your work in Parts a and b.

d. Check your conjecture with that made by other students. Resolve any differences. Then prove the statement upon which you agreed.

29 The linkage shown below is made up of three rhombuses. The longer bars such as $\overline{PQ}$ are three times the lengths of the shorter bars such as $\overline{XP}$.



a. Suppose you hold the linkage fixed at X and copy a shape at Y with pencils inserted at both Z and T. What is the scale factor relating the shape copied at Y to the shape copied at Z? To the shape copied at T?

b. Suppose you fix X and copy a shape at Z with pencils at both Y and T. What is the scale factor relating the shape copied at Z to the shape copied at Y? To the shape copied at T?

c. How can the linkage be used to produce similar shapes with each scale factor below?

i. $\frac{1}{4}$ **ii.** $\frac{3}{4}$

iii. $\frac{3}{2}$ **iv.** $\frac{2}{3}$

d. Make a model of the linkage to check your conclusions in Parts a–c.

e. Design three different linkages that are capable of making a copy similar to the original with scale factor 4.

Review

30 Answer these questions using only mental computation. Then check your work using pencil and paper or a calculator.

a. What number is $\frac{2}{3}$ of 18?

b. What number is $\frac{5}{4}$ of 12.8?

c. 25 is $\frac{5}{4}$ of what number?

d. 10 is $\frac{2}{5}$ of what number?

e. If $x \neq 0$, which of the following ratios are equivalent to $\frac{1}{3}$?

$$\frac{1.2}{3.6} \qquad \frac{12}{4} \qquad \frac{2}{6} \qquad \frac{1}{3x} \qquad \frac{5x}{15x} \qquad \frac{x+6}{x+18} \qquad \frac{x+2}{3x+6}$$

31 For each statement, write the converse of the statement. Then decide if the converse is true or false. If the converse is false, provide a counterexample. If the converse is true, provide reasoning to support your conclusion.

a. If $a > 0$, then $a^2 > 0$.

b. If a is an even number and b is an even number, then ab is a multiple of 4.

c. All quadrilaterals that are rectangles have four right angles.

d. If the linear regression line for two variables has positive slope, then the correlation is positive.

32 Use algebraic reasoning to show that the expressions in each pair are equivalent.

a. $(\sqrt{8k})^2$ and $8k$, for $k > 0$

b. $(12k)^2 + (k\sqrt{5})^2$ and $149k^2$

c. $\dfrac{(5a)^2 - (4a)^2}{2a^2}$ and $\dfrac{9}{2}$

d. $\dfrac{(ab)^2 + 3b^2 - (4ab)^2}{2ab^2}$ and $\dfrac{-15a^2 + 3}{2a}$

e. $\dfrac{5x^3 - 3x^2y + 10x^2}{x^2}$ and $5x - 3y + 10$

33 Solve each equation for the indicated variable.

a. $k = \frac{a}{b}$ for a

b. $\frac{a}{b} = \frac{c}{d}$ for d

c. $x^2 = 36k^2$ for x (assume $k > 0$)

d. $x^2 = y^2 + z^2 - 2yz \cos X$ for $\cos X$

34 Solve each proportion.

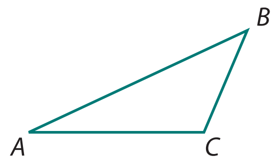
a. $\dfrac{t}{12} = \dfrac{6}{9}$

b. $\dfrac{m+4}{m} = \dfrac{12}{5}$

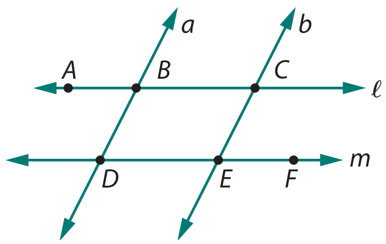
c. $\dfrac{y-3}{10} = \dfrac{2y+5}{3}$

35 Use the provided information to find the measure of each indicated angle.

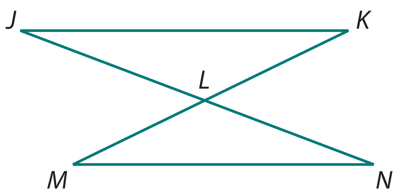
a. If $m\angle A = 25°$ and $m\angle C = 120°$, find $m\angle B$.



b. If $\ell \parallel m$, $a \parallel b$, and $m\angle ABD = 70°$, find $m\angle CEF$.



c. If $m\angle JLK = 123°$, $m\angle LMN = 37°$, and $\overline{JK} \parallel \overline{MN}$, find $m\angle JKL$ and $m\angle MNL$.



36 Rewrite each expression in $ax^2 + bx + c$ form.

a. $(x - 5)(x + 5)$

b. $3(2x + 1)(6 - x)$

c. $x(8 - 3x) + (5x + 3)$

d. $(10x - 6)^2$

37 Find the solution to each inequality. Display your solution two ways, using symbols and using a number line graph.

a. $x^2 + 3x < x + 8$

b. $\frac{20}{x} \geq 10$

c. $2x + 3 > x^2 + 5$

38 Skyler has the responsibility of planning what the free gift will be for people who test drive a car during the month of January. He has been given a budget of \$5,000 for the month.

a. If he wants to have at least 400 gifts available, what is the maximum amount he can spend on each gift?

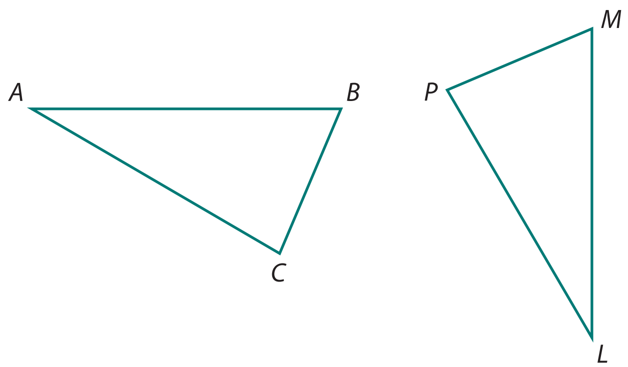
b. If he spends \$18.50 on each gift, what is the maximum number of gifts he can purchase?

c. Write an equation that indicates the relationship between the cost c of the gift and the maximum number g of gifts he can make available.

d. Is this relationship a direct variation relationship, an inverse variation relationship, or neither?

e. Suppose he decides to have two prizes available, a watch that costs \$31.25 and a \$25 gift card to a book store. Draw a graph that indicates all possible combinations of watches and gift cards that he could use.

39 $\triangle ABC \cong \triangle LMP$. Complete each statement about the relationships between the corresponding side lengths and angle measures of the two triangles.



a. $AB =$ _____

b. $MP =$ _____

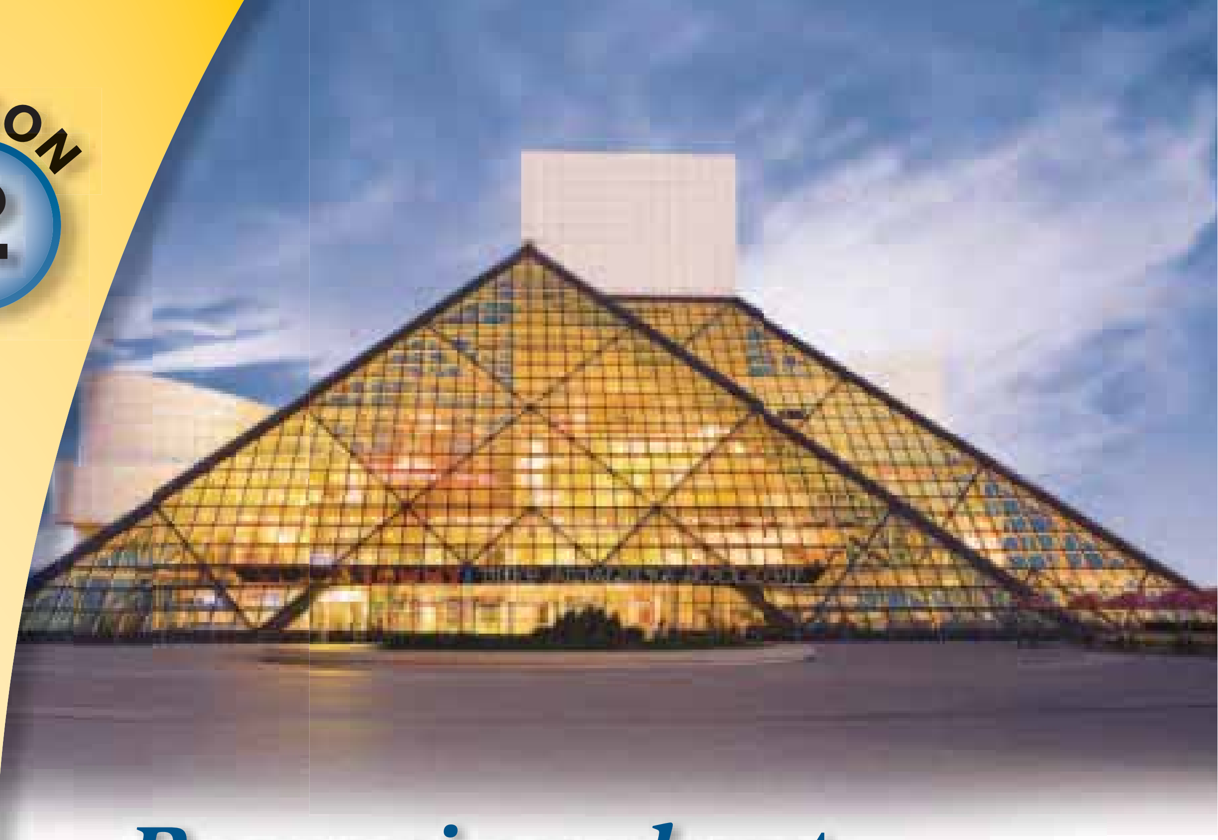
c. $CA =$ _____

d. $m\angle B =$ _____

e. $m\angle A =$ _____

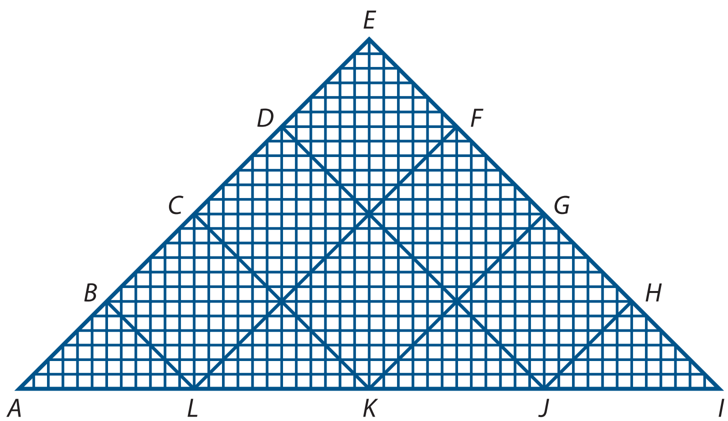
f. $m\angle P =$ _____

LESSON 2



Reasoning about Congruent Triangles

The Rock and Roll Hall of Fame is located in Cleveland, Ohio. The building complex was designed by I.M. Pei to “echo the energy of rock and roll.” A glass pyramidal structure covers the interior exhibits. The front surface design of the “glass tent” is an isosceles triangle with a lattice framework like that shown below. The labeling of points helps to reveal some of the many similar and congruent triangles embedded in the framework.



Think About This Situation

Carefully examine the design of the framework for the front portion of the glass pyramid.

a Identify pairs of triangles that are similar on the basis of the SAS Similarity Theorem.

b Identify pairs of triangles that are similar on the basis of the SSS Similarity Theorem.

c Explain why $\triangle ACK \sim \triangle AEI$ and $\triangle AEI \sim \triangle KGI$.

d Based on the similarity relationships in Part c, why would it make sense to say that $\triangle ACK \sim \triangle KGI$? What is the scale factor relating these two triangles?

e How are $\triangle ACK$ and $\triangle KGI$ related?

f Why are any two congruent triangles also similar? What is the scale factor?

In this lesson, you will explore the relationship between sufficient conditions for congruence of triangles and sufficient conditions for similarity of triangles. You will then use congruence conditions for triangles to solve problems in applied and mathematical contexts.

Investigation 1 Congruence of Triangles Revisited

Congruent triangles have the same shape and size. Your analysis of the Think About This Situation suggests the following equivalent definition of **congruent triangles**.

Two triangles are congruent if and only if they are similar with a scale factor of 1.

In Course 1, you discovered sufficient conditions on side and angle measures to conclude that two triangles are congruent. You assumed those conditions were correct. As you work on the problems of this investigation, look for answers to the following questions:

How are sufficient conditions for congruence of triangles related to sufficient conditions for similarity of triangles?

How is reasoning with congruence conditions for triangles similar to, and different from, reasoning with similarity conditions for triangles?

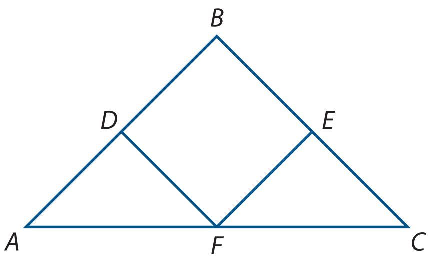
1 Look back at the similarity theorems that you proved in Lesson 1.

a. How would you modify the hypothesis of the SSS Similarity Theorem so that it can be used to show that two triangles are not only similar but also congruent? Write a statement of the corresponding **SSS Congruence Theorem**.

b. Write a theorem for congruence of triangles corresponding to the SAS Similarity Theorem.

c. What needs to be added to the AA Similarity Theorem to make it into a correct statement about conditions that ensure congruence of the triangles? There are two possible different additions. Find them both. Give names to those congruence conditions.

2 The architectural design used in the roof structure of this rest area building along Interstate 94 may remind you of portions of the design of the glass tent of the Rock and Roll Hall of Fame. Similarity, as well as congruence, is suggested visually in both structures.



In the diagram above, $\triangle ABC$ is an isosceles triangle with $\overline{AB} \cong \overline{BC}$. Points D, E, and F are midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively.

a. Identify as many pairs of similar triangles as you can. Then carefully explain how you know they are similar.

b. Identify as many pairs of congruent triangles as you can. Carefully explain how you know they are congruent.

3 For each set of information given below, draw and mark a copy of the diagram. Using only the given conditions, decide if $\triangle ABC \cong \triangle DEF$ or $\triangle ABC \sim \triangle DEF$. Explain your reasoning.

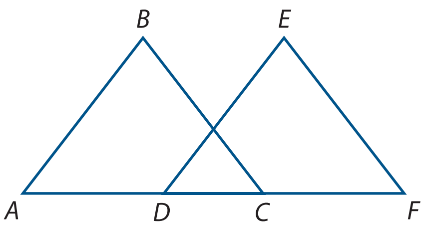
a. $\angle B \cong \angle E$, $\angle A \cong \angle EDF$, and $\overline{AC} \cong \overline{FD}$

b. $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$ and $\overleftrightarrow{BC} \parallel \overleftrightarrow{EF}$

c. $\overline{AD} \cong \overline{FC}$, $\overline{BC} \parallel \overline{EF}$, and $\angle B \cong \angle E$

d. $\overline{BC} \cong \overline{ED}$, $\overline{AB} \cong \overline{FE}$, and $\angle A \cong \angle F$

e. $\overline{AD} \cong \overline{FC}$, $\overline{BC} \parallel \overline{EF}$, and $\overline{AB} \cong \overline{DE}$

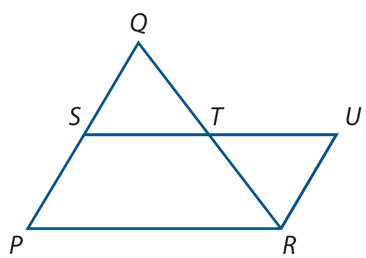


4 In the diagram at the right, S and T are the midpoints of $\overline{PQ}$ and $\overline{QR}$, respectively, and $ST = TU$.

a. Describe a strategy that you would use to prove that $SQ = UR$.

b. Describe a strategy that you would use to prove $ST = \frac{1}{2}PR$.

c. What type of quadrilateral does $PSUR$ appear to be? Describe a strategy that you would use to prove your conjecture.



5 For each part below, draw and label two right triangles.

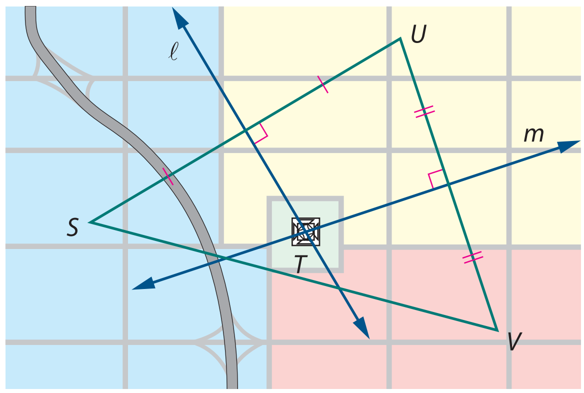
a. What conditions on the hypotenuse and leg of one triangle would guarantee that it is similar to the other triangle? How would you prove your claim?

b. What conditions on the hypotenuse and leg of one triangle would guarantee that it is congruent to the other triangle? How would you prove your claim?

c. What conditions on the hypotenuse and an acute angle of one triangle would guarantee that it is similar to the other triangle? How would you prove your claim?

d. What conditions on the hypotenuse and an acute angle of one triangle would guarantee that it is congruent to the other triangle? How would you prove your claim?

6 Plans for the location of a telecommunications tower that is to serve three northern suburbs of Milwaukee are shown at the right. Design specifications indicate the tower should be located so that it is equidistant from the center S, U, and V of each of the suburbs. In the diagram, line ℓ is the *perpendicular bisector* of $\overline{SU}$. Line m is the perpendicular bisector of $\overline{UV}$. Lines ℓ and m intersect at point T.



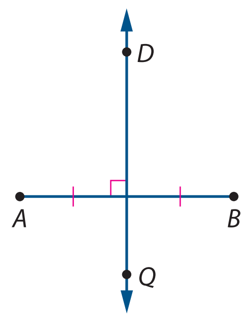
a. On a copy of the diagram, draw $\overline{TS}$ and $\overline{TU}$. Prove that $TS = TU$.

b. Draw $\overline{TV}$ on your diagram. Prove that $TU = TV$.

c. Explain why the tower should be located at point T.

7 Now try to generalize your finding in Problem 6.

a. Is it the case that *any* point on the perpendicular bisector of a line segment is equidistant from the endpoints of the segment? Justify your answer in terms of the diagram at the right.



b. Is it the case that if a point is equidistant from the endpoints of a line segment, then it is on the perpendicular bisector of the segment? Justify your answer in terms of a modified diagram.

c. Summarize your work in Parts a and b by completing this statement.

A point is on the perpendicular bisector of a segment if and only if … .

Summarize the Mathematics

In this investigation, you re-examined the definition of congruence of triangles and sets of minimal conditions sufficient to prove two triangles are congruent.

a Explain why congruence of triangles can be described as a special case of similarity.

b Construct a table that highlights the relationships between the sufficient conditions for similarity and sufficient conditions for congruence of triangles.

c Knowing that two pairs of corresponding angles in two triangles are congruent ensures that the triangles are similar, but it does *not* ensure that they are congruent. Why?

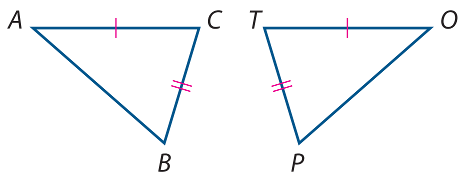
d How were conditions for congruence of triangles used in your justification in Problem 7?

Be prepared to explain your ideas to the class.

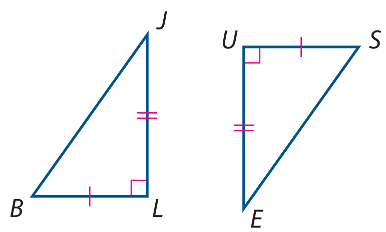
Check Your Understanding

Examine each of the following pairs of triangles and their markings showing congruence of corresponding angles and sides. In each case, decide whether the information given by the markings ensures that the triangles are congruent. If the triangles are congruent, write a congruence relation showing the correspondence between vertices. Cite an appropriate congruence theorem to support your conclusion.

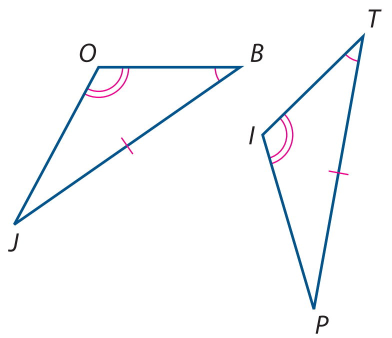
a.



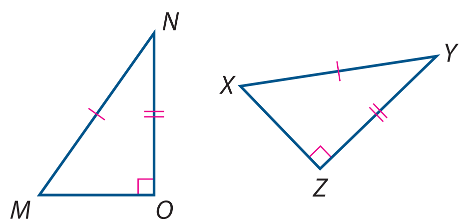
b.



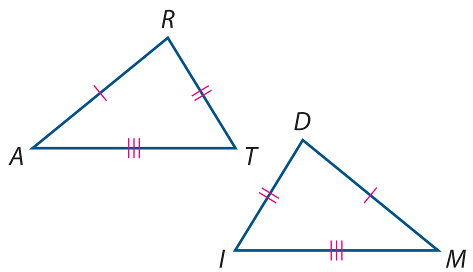
c.



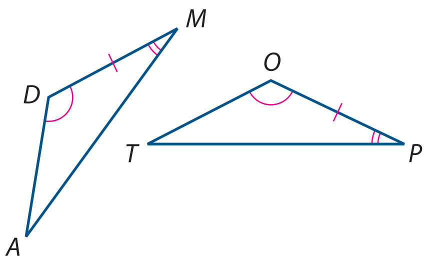
d.



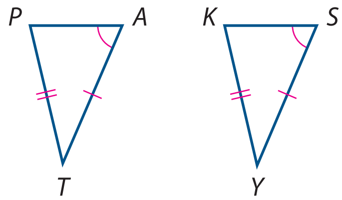
e.



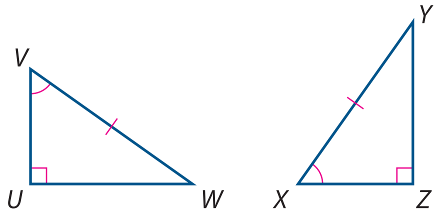
f.



g.

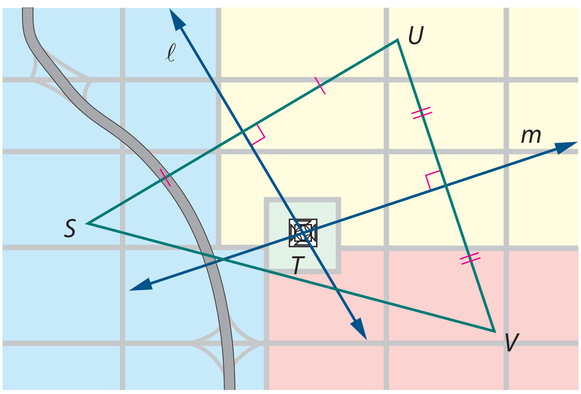


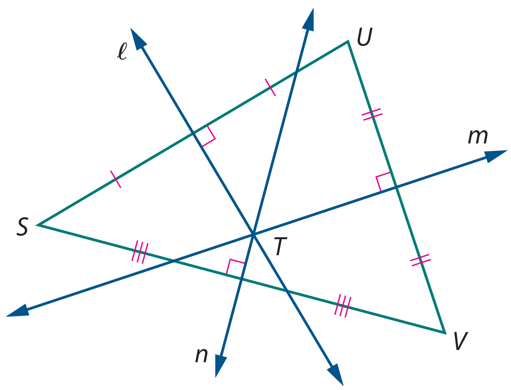
h.



Investigation 2 Congruence in Triangles

In Investigation 1, you located a telecommunications tower that was to be equidistant from the centers of three Milwaukee suburbs. The strategy involved considering the triangle formed by the centers of the suburbs, drawing perpendicular bisectors of two of its sides, and then reasoning with congruent triangles.





If the perpendicular bisector n of the third side of $\triangle SUV$ is drawn, it appears that the three lines are **concurrent**. That is, they intersect at a common point. Will this be the case for other triangles?

As you work on the problems in this investigation, look for answers to these questions:

Under what conditions will the perpendicular bisectors of the sides of a triangle be concurrent?

Under what conditions will the bisectors of the angles of a triangle be concurrent?

Under what conditions will the medians of a triangle be concurrent?

What are special properties of these points of concurrency?

1 **Focusing on Perpendicular Bisectors of Sides** Look more closely at your solution of the telecommunications tower location problem. Refer to the diagram on the lower right of the previous page.

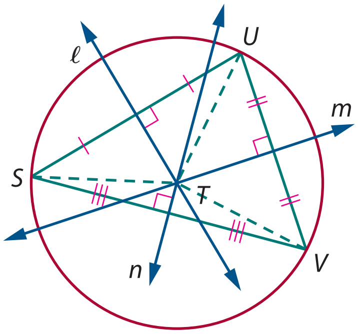
a. If $\ell \perp \overline{SU}$ and $m \perp \overline{UV}$, then ℓ and m must intersect at a point T as shown. Why is it not possible that $\ell \parallel m$?

b. On a copy of the diagram, draw in a second color $\overline{TS}$, $\overline{TU}$, and $\overline{TV}$. Discuss with classmates how you used congruent triangles to show that $TU = TV$. That $TU = TS$.

c. From Part b, you were able to conclude that $TS = TU = TV$ and therefore a telecommunications tower located at point T would be equidistant from points S, U, and V. Why must point T be on line n, the perpendicular bisector of $\overline{SV}$?

d. Your work in Parts a–c proves that the perpendicular bisectors of the sides of $\triangle SUV$ are concurrent. $\triangle SUV$ is an acute triangle. Draw a diagram for the case where $\triangle SUV$ is a right triangle. Where $\triangle SUV$ is an obtuse triangle. Explain why the reasoning in Parts a–c applies to any triangle.

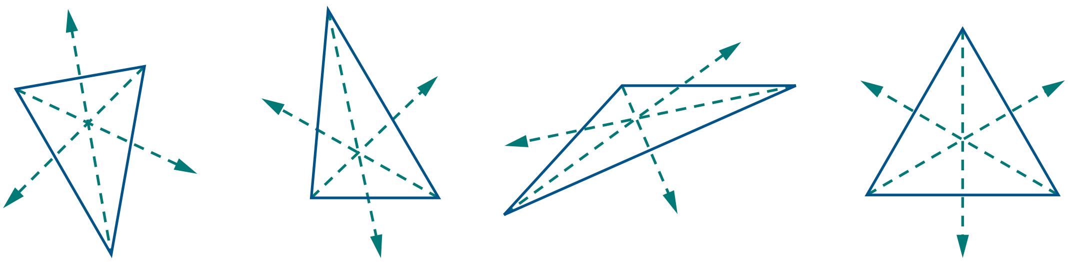
2 The point of concurrency of the perpendicular bisectors of the sides of a triangle is called the **circumcenter** of the triangle.



a. Why can a circle with center T and radius $\overline{TS}$ be *circumscribed* about $\triangle SUV$ as shown?

b. Explain with an illustration why a circle can be circumscribed about *any* triangle.

3 **Focusing on Angle Bisectors** In each of the four triangles below, the angle bisectors at each vertex have been constructed.



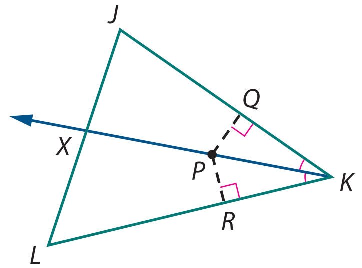
a. What appears to be true about the three angle bisectors in each case?

b. Use interactive geometry software or a compass and straightedge to test your ideas in the case of other triangles. Compare your findings with those of your classmates.

c. If you pick *any* point on the bisector of one of the angles, how does that point seem to be related to the sides of the angle? Compare your ideas with other students. Resolve any differences.

To find logical explanations for your observations in Problem 3, it is helpful to consider the angle bisectors of a triangle, one angle bisector at a time.

4 In the diagram at the right, $\overrightarrow{KX}$ is the bisector of $\angle JKL$ and P is a point on $\overrightarrow{KX}$. $\overline{PQ} \perp \overleftrightarrow{JK}$ and $\overline{PR} \perp \overleftrightarrow{KL}$. The **distance from a point to a line** is the length of the perpendicular segment from the point to the line.



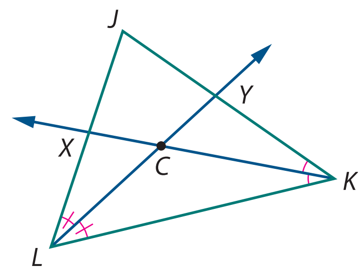
a. Prove that point P is equidistant from the sides of $\angle JKL$ by showing $PQ = PR$.

b. Is it also the case that if a point is equidistant from the sides of an angle, then it is on the bisector of the angle? Justify your answer in terms of a modified diagram.

c. Summarize your work in Parts a and b by completing this statement.

A point is on the bisector of an angle if and only if … .

5 The diagram in Problem 4 is reproduced at the right with $\overrightarrow{LY}$ the bisector of $\angle JLK$ added to the diagram.



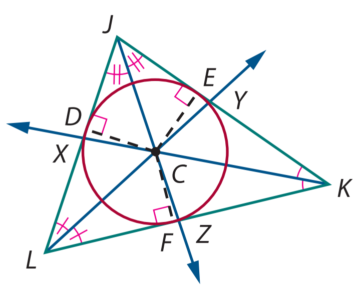
a. Why must angle bisectors $\overrightarrow{KX}$ and $\overrightarrow{LY}$ intersect at some point C?

b. How is point C related to the sides of $\angle JKL$? The sides of $\angle JLK$?

c. Why must point C be on the bisector of $\angle LJK$?

d. Your work in Parts a–c proves that the bisector of the angles of $\triangle JKL$ are concurrent. $\triangle JKL$ is an acute triangle. Draw a diagram for the case where $\triangle JKL$ is a right triangle. Where $\triangle JKL$ is an obtuse triangle. Explain why the reasoning in Parts a–c applies to *any* triangle.

6 The point of concurrency of the bisectors of the angles of a triangle is called the **incenter** of the triangle.



a. Why can a circle with center C and radius $\overline{CD}$ be *inscribed* in $\triangle JKL$ as shown?

b. Explain with an illustration why a circle can be inscribed in *any* triangle.

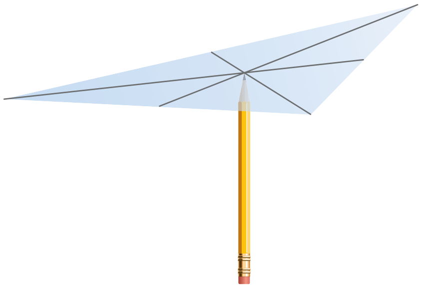
7 **Focusing on Medians** You have now been able to prove that the three perpendicular bisectors of the sides of a triangle are concurrent and so are the three angle bisectors. Now investigate to see if the three medians of a triangle are concurrent.

a. Use interactive geometry software or a compass and straightedge to draw a triangle. Then construct the three medians. What appears to be true?

b. Test your observations in the case of other triangles. Compare your findings with those of your classmates. Then write a statement summarizing your findings. In Extensions Task 30, you are asked to prove your claim.

8 The point of concurrency of the medians of a triangle is called the **centroid** of the triangle.

- Cut a triangle from poster board or cardboard.
- Find and mark the centroid of your triangle.
- Verify that the centroid is the *center of gravity* of the triangle by balancing the triangle on the tip of a pencil or pen.



Summarize the Mathematics

In this investigation, you explored centers of triangles and their properties.

a Under what conditions will the three perpendicular bisectors of the sides of a triangle be concurrent? The three angle bisectors? The three medians?

b How can you locate the center of a circle inscribed in a triangle?

c How can you locate the center of a circle circumscribed about a triangle?

d How can you locate the center of gravity of a triangular shape of uniform density?

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

In the diagram, $\triangle ABC$ is an equilateral triangle. Point P is the circumcenter of $\triangle ABC$.

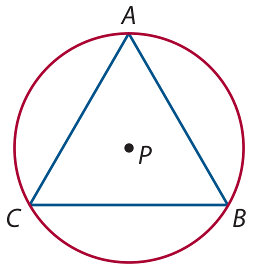
a. Use congruent triangles to prove that point P is also the incenter of $\triangle ABC$.

b. Prove that point P is also the centroid of $\triangle ABC$.

c. Suppose $AB = 12$ cm.

  i. Find the length of the radius of the circumscribed circle.

  ii. Find the length of the radius of the inscribed circle.

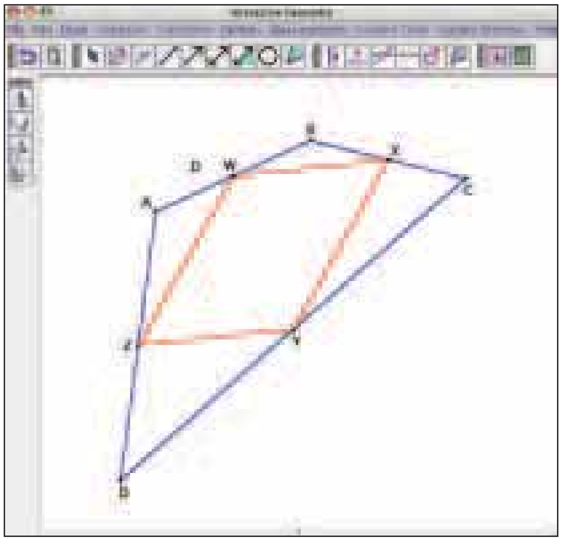


Investigation 3 Congruence in Quadrilaterals

In Investigations 1 and 2, you saw that strategies for solving problems and proving claims using congruent triangles were similar to strategies you used in Lesson 1 when reasoning with similar triangles. You identified triangles that could be proven congruent and then used the congruence of some corresponding sides or angles to draw desired conclusions.

In this investigation, you will extend those reasoning strategies to solve more complex problems by drawing on your knowledge of relations among parallel lines. For example, when the midpoints of the sides of a quadrilateral are connected in order, a new quadrilateral is formed.

CPMP-Tools

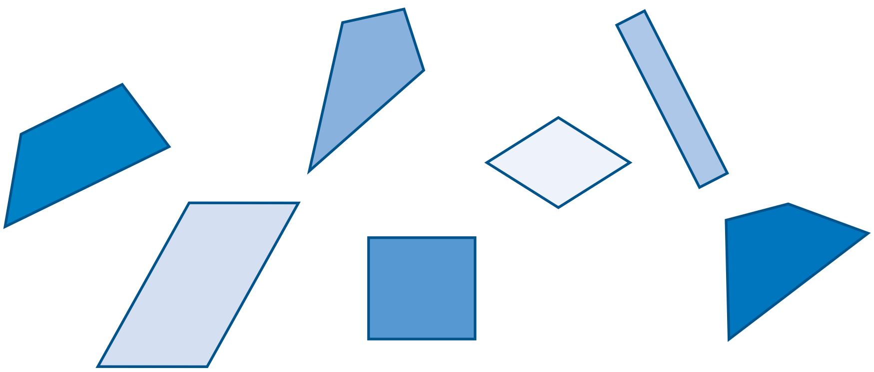


What kind of special quadrilateral is formed?
Will that always be the case?

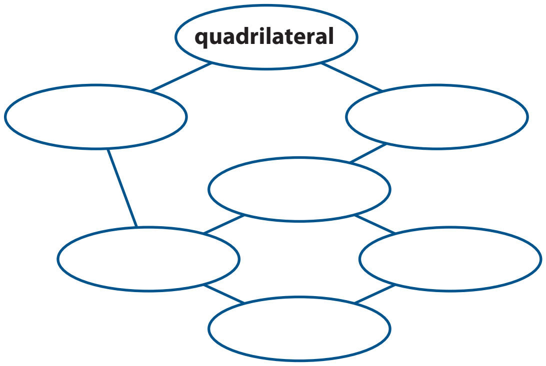
As you work on the problems of this investigation, look for answers to those questions and the more general questions:

How can you use congruent triangles to establish properties of special quadrilaterals, and what are those properties?

1. Quadrilaterals come in a variety of specialized shapes. There are trapezoids, kites, parallelograms, rhombuses, rectangles, and squares, as well as quadrilaterals with no special additional characteristics.



a. Complete a copy of the *quadrilateral tree* shown below by filling in each oval with the name of one of the six special quadrilaterals listed. If the oval for a quadrilateral is connected to one or more ovals *above* it, then that quadrilateral must also have the properties of the quadrilateral(s) in the ovals above it to which it is connected.



b. Suppose you know that a characteristic X (such as opposite sides congruent) is true for parallelograms. For which other quadrilaterals must characteristic X also be true? Explain your reasoning.

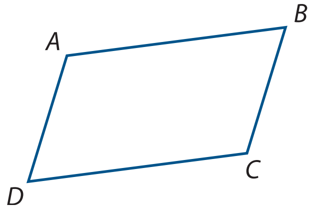
c. Suppose a characteristic Y is true for all kites. Which other quadrilaterals must also have characteristic Y?

2 In your work in Course 1 and Course 2, you showed that the following two statements are equivalent. That is, if you accepted one statement as the definition of a parallelogram, you could prove the other statement as a theorem.

I. *A parallelogram is a quadrilateral that has two pair of opposite sides congruent.*

II. *A parallelogram is a quadrilateral that has two pair of opposite sides parallel.*

Using the diagram at the right and your knowledge of congruent triangles and properties of parallel lines:



a. explain how you could prove Statement II using Statement I.

b. explain how you could prove Statement I using Statement II.

3 **Properties of Parallelograms** A class at Rockbridge High School was asked to make a list of properties of *every* parallelogram, and students prepared the following list. Working in groups, examine each statement to see if you think it is correct. If you think it is correct, construct a proof. If you think it is incorrect, give a counterexample. Share the work. Be prepared to explain your conclusions and reasoning to the entire class.

If a quadrilateral is a parallelogram, then:

a. each diagonal divides the parallelogram into two congruent triangles.

b. opposite angles are congruent.

c. consecutive angles are supplementary.

d. the diagonals are congruent.

e. the diagonals bisect each other.

f. the diagonals are perpendicular.

4 How would your answers to Problem 3 change if "parallelogram" was replaced by:

a. "rectangle"?

b. "rhombus"?

c. "kite"?

5 How are your responses to Problem 4 reflected in the quadrilateral tree you completed for Problem 1 Part a?

6 **Sufficient Conditions for Parallelograms** Here are additional conditions that may or may not ensure that a quadrilateral is a parallelogram. Investigate each conjecture by drawing shapes satisfying the conditions and checking to see if the quadrilateral is a parallelogram. Share the workload among members of your group. Compile a group list of conditions that seem to guarantee that a quadrilateral is a parallelogram. Be prepared to explain your conclusions and reasoning to the entire class.

a. If a diagonal of a quadrilateral divides it into two congruent triangles, then the quadrilateral is a parallelogram.

b. If a quadrilateral has two pairs of opposite angles congruent, then it is a parallelogram.

c. If a quadrilateral has one pair of opposite sides parallel and the other pair of opposite sides congruent, then it is a parallelogram.

d. If a quadrilateral has two distinct pairs of consecutive sides congruent, then it is a parallelogram.

e. If a quadrilateral has one pair of opposite sides congruent and parallel, then it is a parallelogram.

7 Now refer back to the interactive geometry software screen and questions posed at the beginning of this investigation. Compare the conjecture that you and your classmates made with the statement below.

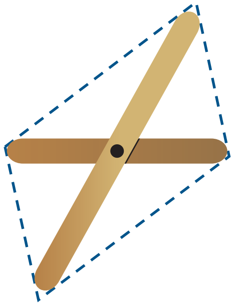
If the midpoints of consecutive sides of any quadrilateral are connected in order, the resulting quadrilateral is a parallelogram.

a. Draw and label a diagram representing this statement. Then add a diagonal of the original quadrilateral to your diagram.

b. Write an argument to prove this **Midpoint Connector Theorem for Quadrilaterals**.

c. What was the key previously proven result on which your proof depended?

8 **Diagonal Connections** In your earlier work, you represented the diagonals of a quadrilateral with two linkage strips attached at a point. Complete the following table that relates conditions on diagonals to special quadrilaterals. Be prepared to explain how you can use congruent triangles to help justify your conclusion.



If the diagonals of a quadrilateral … ,	then the quadrilateral is a … .
a. bisect each other	____________
b. bisect each other and are equal in length	____________
c. are perpendicular bisectors of each other	____________
d. are the same length and are perpendicular bisectors of each other	____________
e. are such that one diagonal is the perpendicular bisector of the other	____________

Summarize the Mathematics

In this investigation, you explored how conditions for congruence of triangles could be used to prove properties of special quadrilaterals and sufficient conditions for a quadrilateral to be a parallelogram or a special parallelogram.

a Rectangles are a special type of parallelogram. List all the properties they have in common. Identify at least two properties of rectangles that are *not* properties of all parallelograms.

b A square is a special type of rhombus. What properties do they have in common? Identify at least two properties of squares that are *not* properties of all rhombuses.

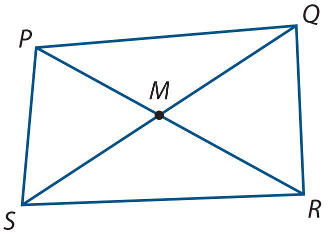
c Summarize your work in Problems 3, 4, and 8 by writing if-and-only-if statements characterizing special quadrilaterals in terms of relationships involving their diagonals.

d What strategies are helpful in using congruence to establish properties of special quadrilaterals?

Be prepared to share your lists and reasoning with the class.

Check Your Understanding

For each set of conditions, explain why they do or do not ensure that quadrilateral $PQRS$ is a parallelogram.



a. $\angle SPQ \cong \angle QRS$ and $\angle PQR \cong \angle PSR$

b. $\overline{PQ} \parallel \overline{RS}$ and $\overline{QR} \cong \overline{PS}$

c. $\overline{PQ} \cong \overline{RS}$ and $\overline{QR} \cong \overline{PS}$

d. $\overline{PQ} \cong \overline{QR}$ and $\overline{RS} \cong \overline{PS}$

e. M is the midpoint of $\overline{PR}$ and of $\overline{QS}$.

Investigation 4 Congruence-Preserving Transformations

In Course 2, you studied transformations of a plane defined by coordinate rules. You explored translations, rotations about the origin, and the line reflections across the x- and y-axes, lines parallel to the axes, and lines with equations $y = x$ and $y = -x$. Using the distance formula, you were able to prove that these transformations preserve distances and therefore images of transformed figures are congruent to the original figures. You used these *congruence-preserving transformations* to reposition figures on a computer screen and create interesting animations.

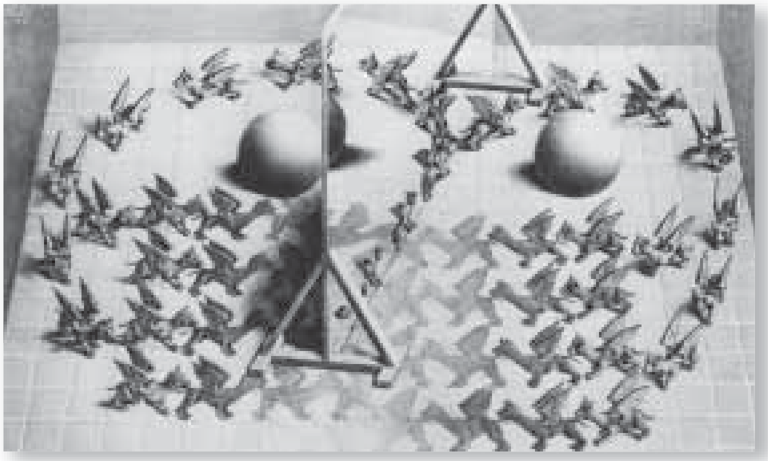
In this investigation, you will explore properties and applications of these transformations without the use of coordinates and without restrictions on the positions of the lines of reflection or the centers of rotation. Once again, congruence of triangles will be a key tool in your work.

As you work on the problems in this investigation, look for answers to these questions:

What are the connections between line reflections and translations and rotations?

How can you prove properties of these congruence-preserving transformations without the use of coordinates and how can you use those properties to solve problems?

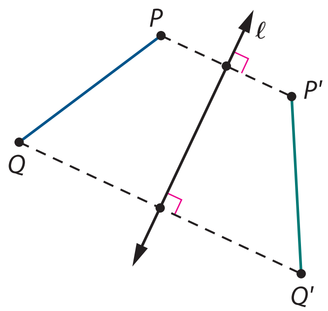
Reasoning with Line Reflections In Escher's *Magic Mirror* shown below, figures and their reflected images reside in the same plane. The relationship between a figure and its reflected image agrees with the following definition of a line reflection.



Source: *M. C. Escher, His Life and Complete Graphic Works* © 1981

A **reflection across line ℓ** is a transformation that maps each point P of the plane onto an image point P' as follows:

- If point P is not on ℓ, then ℓ is the perpendicular bisector of $\overline{PP'}$.
- If point P is on ℓ, then $P' = P$. That is, P is its own image.



1 The diagram above illustrating the definition of a line reflection suggests that a *line segment and its reflected image are the same length*. That is, $PQ = P'Q'$. Parts a–d will help you justify that conclusion for any choice of points P and Q.

a. On a copy of the diagram, draw $\overline{PP'}$ and $\overline{QQ'}$. Label the points of intersection with line ℓ, M and N, respectively. Draw $\overline{QM}$ and $\overline{Q'M}$.

b. Use reasoning with congruent triangles to prove that $PQ = P'Q'$.

c. In the case above, points P and Q are on the same side of the line of reflection. Draw three diagrams illustrating other possible positions of points P and Q relative to line ℓ.

d. Discuss with classmates how you could prove that $PQ = P'Q'$ in each of the cases in Part c.

M.C. Escher Foundation

2 Draw a line ℓ and a triangle ABC.

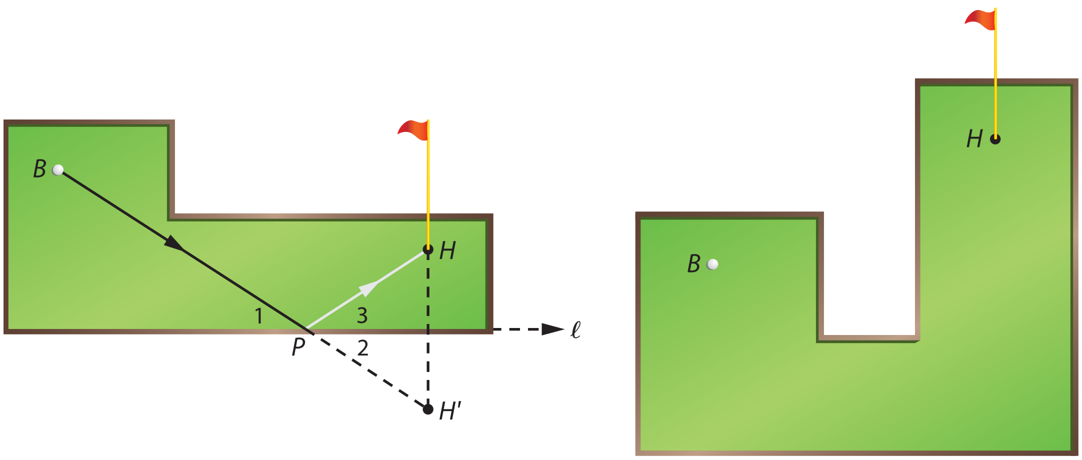
a. Find the reflected image of $\triangle ABC$ across line ℓ. Label the image $\triangle A'B'C'$.

b. Use the result of Problem 1 to prove that $\triangle A'B'C' \cong \triangle ABC$.

c. How could you justify the claim that an *angle and its reflected image across a line are congruent*?

Playing Smarter When a ball with no spin and medium speed is banked off a flat surface, the angles at which it strikes and leaves the surface are congruent. You can use this fact and knowledge of line reflections to your advantage in games of miniature golf and billiards.

3 To make a hole-in-one on the miniature golf green on the left below, visualize point H', the reflected image of the hole H across line ℓ. Aim for the point P where $\overline{BH'}$ intersects ℓ.

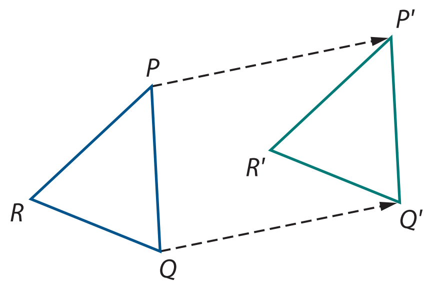


a. If you aim for point P, give reasons to justify that the ball will follow the indicated path to the hole. That is, show $\angle 1 \cong \angle 3$.

b. The diagram at the right shows another par 2 green. On a copy of that diagram, test if you can make a hole-in-one using a single line reflection as in Part a.

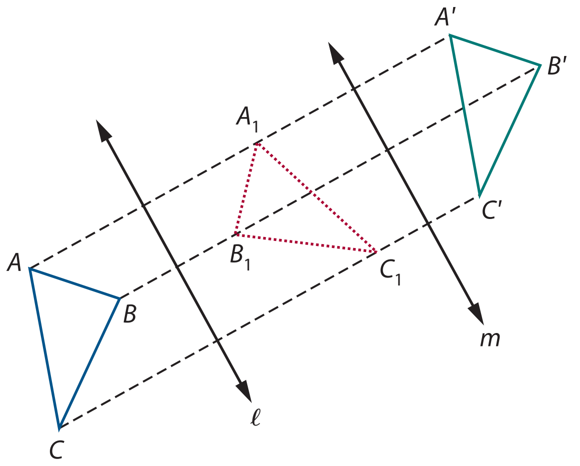
c. Is there a way that by finding the reflection image of the hole H across one side and then the reflection of that image across another side, that you can find a point to aim and still make a hole-in-one? Illustrate your answer. Explain why a hit golf ball will follow the path you have drawn.

The Reflection-Translation Connection Recall that a **translation** is a transformation that "slides" all points in the plane the same distance and same direction. That is, if points P' and Q' are the images of points P and Q under a translation, then $PP' = QQ'$ and $\overline{PP'} \parallel \overline{QQ'}$.



In Problems 4 and 5, you will investigate connections between translations and line reflections.

4 The diagram below shows the image of $\triangle ABC$ under a composition of two line reflections, first across line ℓ (image $\triangle A_1B_1C_1$) and then across line m, where $\ell \parallel m$ (image $\triangle A'B'C'$).



a. Explain why $\triangle A'B'C' \cong \triangle ABC$.

b. How does $\triangle A'B'C'$ appear to be related to $\triangle ABC$ by position?

c. What kind of special quadrilateral is $AA'C'C$? Justify your claim.

d. Explain why $AA' = CC'$ and $\overline{AA'} \cong \overline{CC'}$.

e. Is it also the case that $BB' = CC'$ and $\overline{BB'} \parallel \overline{CC'}$? Justify your answer.

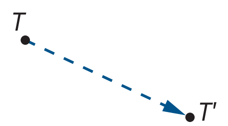
f. Suppose P is a point on ℓ and P' is the image of P under the composition of the two reflections. Draw a diagram. Explain why it is also the case that $PP' = AA'$ and $\overline{PP'} \parallel \overline{AA'}$.

5 Look back at your work in Parts d–f of Problem 4.

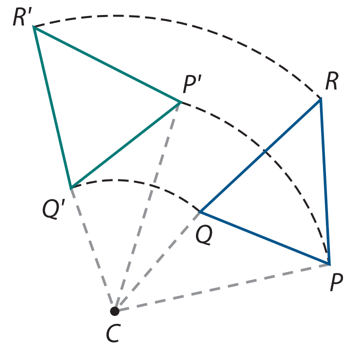
a. Explain why the composition of two reflections across parallel lines is a translation.

b. How would you describe the *magnitude* (distance points are translated) of the translation? The direction?

c. In the diagram at the right, point T' is the translated image of point T. On a copy of the diagram, find two lines so that the composition of reflections across those lines maps T to T'. Compare the two lines you found with those found by other students. Resolve any differences.

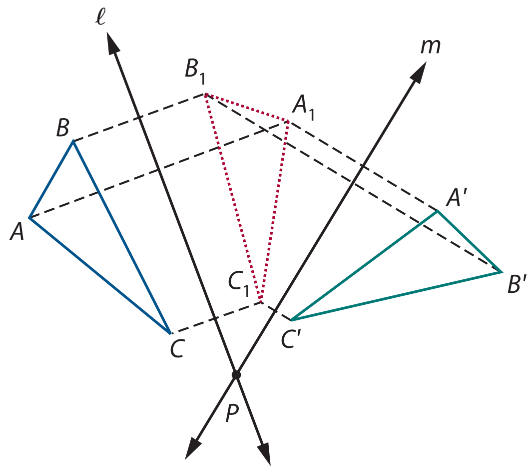


The Reflection-Rotation Connection Recall that a **rotation** is a transformation that "turns" all points in the plane about a fixed center point through a specified angle. That is, if points P' and Q' are the images of points P and Q under a counterclockwise rotation about point C, then $CP = CP'$, $CQ = CQ'$, and $m\angle PCP' = m\angle QCQ'$.



In Problems 6 and 7, you will investigate connections between rotations and line reflections.

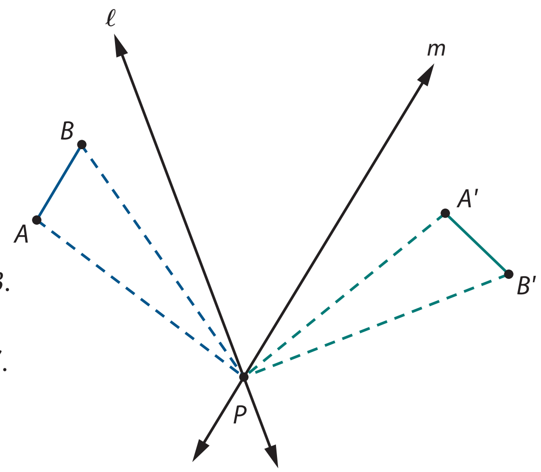
6 The diagram below shows $\triangle A'B'C'$ the image of $\triangle ABC$ under a composition of two reflections across intersecting lines, first a reflection across ℓ and then a reflection across m.



a. Explain why $\triangle A'B'C' \cong \triangle ABC$.

b. How does $\triangle A'B'C'$ appear to be related to $\triangle ABC$ by position?

c. A portion of the above diagram is reproduced here.



i. Explain why $PA = PA'$ and $PB = PB'$.

ii. Explain why $m\angle A'PB' = m\angle APB$.

iii. Explain why $m\angle APA' = m\angle BPB'$.

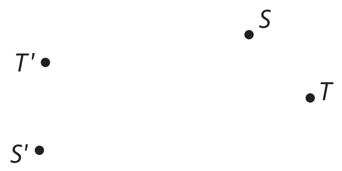
d. Suppose X is a point on ℓ ($X \neq P$) and X' is the image of X under the composition of a reflection across line ℓ followed by a reflection across line m. Draw a diagram. Explain why it is also the case that $m\angle XPX' = m\angle APA'$.

7 Look back at your work in Parts c and d of Problem 6.

a. Explain why the composition of reflections across two intersecting lines is a rotation about the point of intersection.

b. How would you describe the *directed angle* of the rotation?

c. In the diagram below, points S' and T' are the images of points S and T under a rotation about a point C.



On a copy of the diagram, find two lines so that the composition of reflections across those lines maps S to S' and T to T'. You might start by finding a line ℓ so that the reflection of point S across ℓ is point S'.

d. What is the center and directed angle of the rotation?

Summarize the Mathematics

In this investigation, you re-examined line reflections, translations, and rotations from a synthetic perspective and explored how congruence of triangles could be used to establish properties of those transformations.

a Suppose in a diagram, $\triangle M'N'O'$ is the reflected image of $\triangle MNO$ across a line ℓ.

- **i.** How could you find the line of reflection?
- **ii.** How are side lengths and angle measures of the two triangles related?

b Under what condition is the composition of two line reflections a translation? How can you determine the direction and magnitude of the translation?

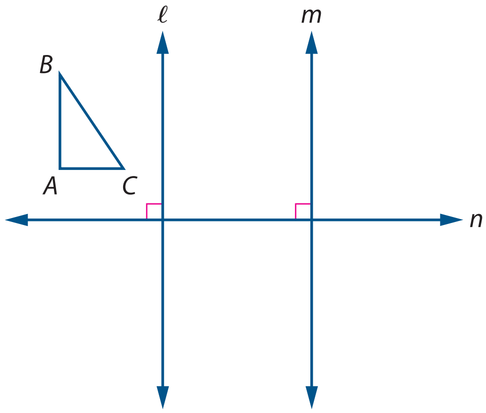
c Under what condition is the composition of two line reflections a rotation? Where is the center of rotation? How can you determine the measure of the angle of rotation?

d You proved that line reflections preserve segment lengths and angle measures and map a triangle onto a congruent triangle. Why must translations and rotations have these same properties?

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

Using a copy of the diagram below, draw the image of $\triangle ABC$ under the composition of successive reflections across line ℓ, line m, and line n. Label the final image triangle $\triangle A'B'C'$.



a. Explain why $\triangle A'B'C' \cong \triangle ABC$.

b. Compare the clockwise/counterclockwise orientation of the corresponding vertices of the two triangles. What explains this fact?

c. How would you describe this transformation?

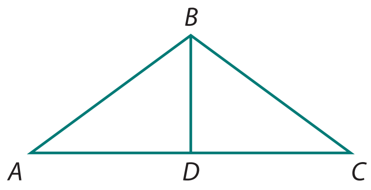
d. Draw segments $\overline{AA'}$ and $\overline{BB'}$. What appears to be true about the midpoints of those segments? Is the same true for the midpoint of $\overline{CC'}$?

e. How could you use Problem 6 on page 177 to prove that your observation in Part d is true for any point P and its image P' under this composite transformation? Start by drawing a new diagram showing lines ℓ, m, and n. Mark a point P and locate its image P'.

On Your Own

Applications

1 Prove, if possible, that $\triangle ABD \cong \triangle CBD$ under each set of conditions below.



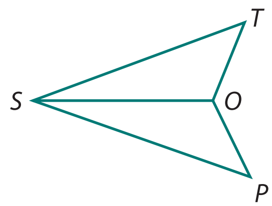
a. $\overline{AB} \cong \overline{BC}$ and $\angle ABD \cong \angle CBD$.

b. $\overline{BD}$ is the perpendicular bisector of $\overline{AC}$.

c. $\overline{BD} \perp \overline{AC}$ and $m\angle A = m\angle C$.

d. $AB = BC$ and D is the midpoint of $\overline{AC}$.

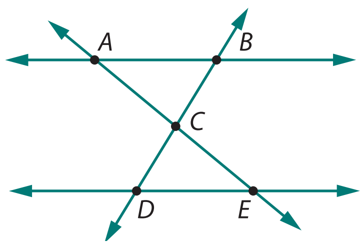
2 Suppose you are given the following information about the figure shown at the right.



$$\overline{ST} \cong \overline{SP} \text{ and } \overline{TO} \cong \overline{PO}$$

Can you conclude that $\triangle STO \cong \triangle SPO$? Write a proof or give a counterexample.

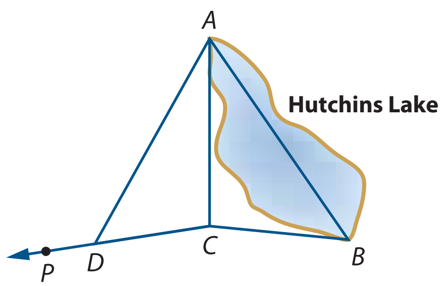
3 For the four lines shown, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$ and C is the midpoint of $\overline{BD}$.



a. Prove that $\triangle ABC \cong \triangle EDC$.

b. Using your work in Part a, explain why you also can conclude that C is the midpoint of $\overline{AE}$.

4 Hutchins Lake is a long, narrow lake. Its length is represented by $\overline{AB}$ in the diagram shown below. Dmitri designed the following method to determine its length. First, he paced off and measured $\overline{AC}$ and $\overline{BC}$. Then, using a transit, he made $m\angle PCA = m\angle ACB$. He then marked point D on $\overrightarrow{CP}$ so that $DC = BC$, and he measured $\overline{AD}$.



a. Dmitri claimed $AB = AD$. Is he correct? Justify your answer.

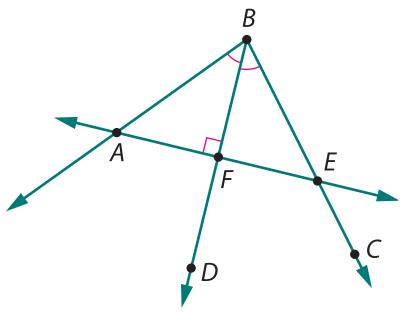
b. Andrea claimed she could find the length AB of the lake without sighting $\triangle ADC$. Suppose $AC = 500$ m, $BC = 200$ m, and $m\angle ACB = 100°$. What is the length of Hutchins Lake?

5 Consider the diagram below, in which $\overrightarrow{BD}$ is the angle bisector of $\angle ABC$ and $\overleftrightarrow{AE}$ is perpendicular to $\overrightarrow{BD}$.

a. Write a proof for each statement.

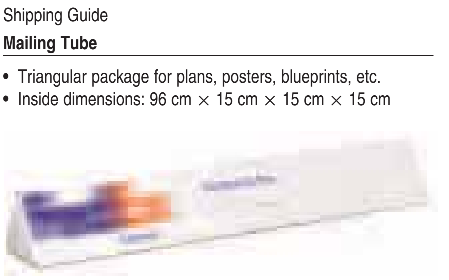
i. $\triangle ABF \cong \triangle EBF$

ii. $\triangle ABE$ is an isosceles triangle.



b. On a copy of the diagram, draw a line perpendicular to $\overrightarrow{BD}$ through point C and intersecting $\overleftrightarrow{AB}$. Identify all pairs of congruent and similar triangles. Justify your response.

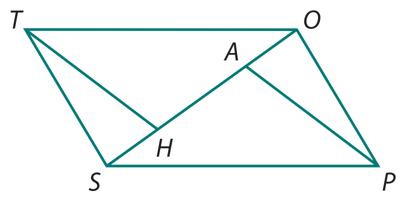
6 To ship cylindrical objects such as plans, posters, and blueprints, FedEx supplies packages in the shape of an equilateral triangular prism.



a. Sketch the triangular base of the prism and the circular outline of a rolled up blueprint of largest diameter that can be shipped in this package.

b. What is the diameter of the rolled up blueprint in Part a?

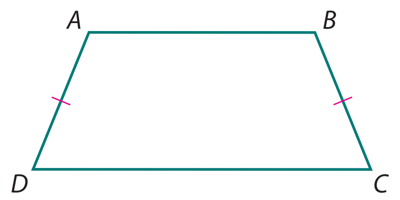
7 In the diagram, $\overline{TO} \cong \overline{SP}$, $\overleftrightarrow{TO} \parallel \overleftrightarrow{SP}$, and $\overleftrightarrow{TH} \parallel \overleftrightarrow{PA}$.



a. Prove that $\overline{TH} \cong \overline{PA}$.

b. Prove that $\triangle STH \cong \triangle OPA$.

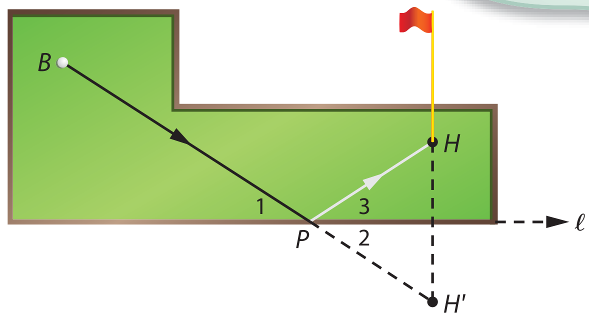
8 Trapezoid $ABCD$ is an **isosceles trapezoid**—it has exactly one pair of parallel sides and the nonparallel sides are congruent. Write an argument to prove that the base angles, $\angle D$ and $\angle C$, are congruent.



9 On a copy of the diagram in Applications Task 8, draw in diagonals $\overline{AC}$ and $\overline{BD}$. What appears to be true about the two diagonals? Write an argument that proves that your observation is correct.

10 At the right is the par 2 miniature golf green reproduced from Investigation 4. You were able to show that if point H' is the reflection image of the hole H across side ℓ, then a putt aimed at point P will produce the indicated path of the ball.

The path of the ball has another important property that has physical applications. Of all paths from point B to line ℓ to point H, the path $\overline{BP} + \overline{PH}$ has minimal length.

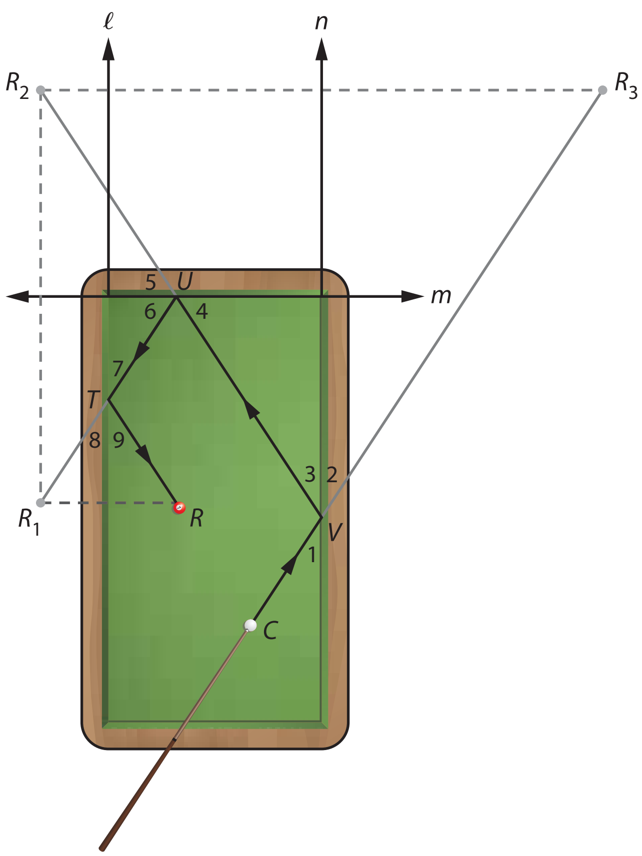


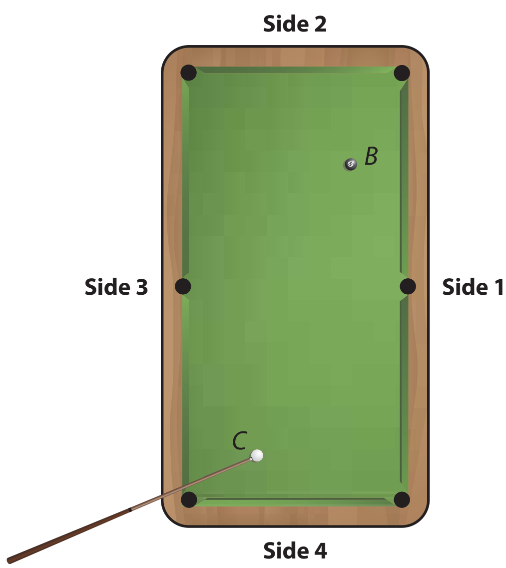
a. On a copy of the diagram, pick a different point Q on ℓ. Draw the path from B to Q and then from Q to H. Show that this path is longer than the path $\overline{BP} + \overline{PH}$.

b. Use the Triangle Inequality to prove that for any point X on ℓ, $X \neq P$, $BX + XH > BP + PH$.

11 The strategies you used in making a hole-in-one in miniature golf can also be applied in games such as pool and billiards.

a. On the billiards table below, a player wanted to strike the red ball by bouncing the cue ball off three cushions as shown. She visualized the reflection of the red ball R across side ℓ, then visualized the reflection of the image R_1 across side m. Finally, she visualized the reflection of the image R_2 across side n. She aimed for point V where $\overline{CR_3}$ intersects side n, being careful to exert enough force and not put any spin on the ball. Explain as precisely as you can why the cue ball will follow the indicated path and strike the red ball.





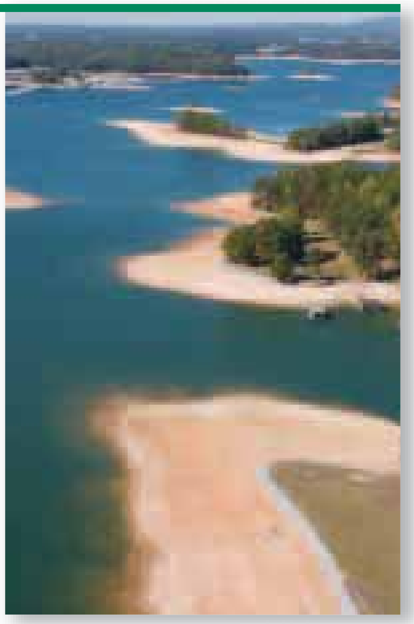
b. On copies of the pool table, show how to determine the path of the cue ball *C* so that it bounces off the given sides and then hits the eight ball *B*.

i. side 1

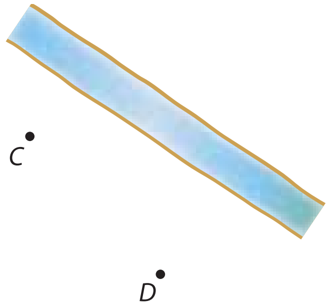
ii. side 2

iii. side 1, then side 2

iv. side 3, then side 2, then side 1



12 In 2007, the southeastern and southwestern regions of the United States suffered severe drought due in part to climate changes. Some communities turned to rivers as a source of water. But pumping water from a river can be expensive. Two small, rural communities along the Flint River in Georgia decided to pool their resources and build a pumping station that would pipe water to both communities.

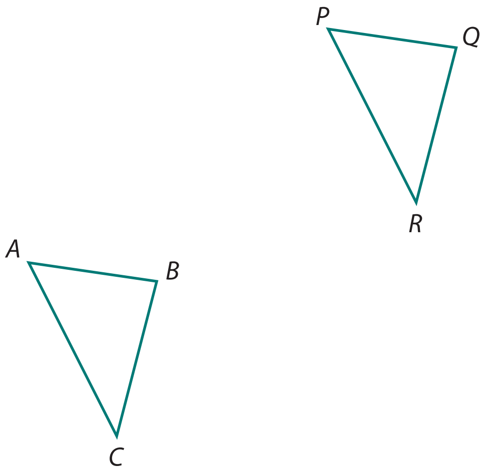


On a copy of the diagram, determine a location of the pumping station that will use the minimum amount of pipe. Then explain why the amount of pipe required is the minimum possible.

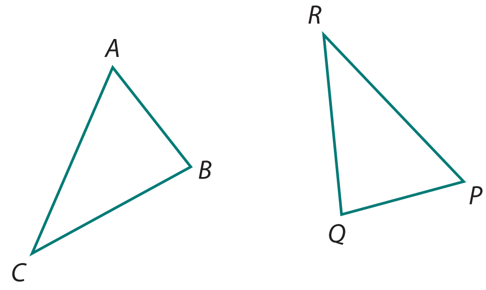
13 In each case below, $\triangle ABC \cong \triangle PQR$.

- On a copy of each pair of triangles, determine if you can map $\triangle ABC$ onto $\triangle PQR$ by a composition of line reflections. If so, find and label the appropriate line(s).
- In each case, identify the composite transformation as precisely as you can. If it is a rotation, give its center and directed angle of rotation. If it is a translation, give its magnitude and direction.

a.



b.



Connections

14 In Investigation 1, you used conditions for similarity of triangles and a scale factor of 1 to establish four sets of conditions that are sufficient to prove that two triangles are congruent: SSS, SAS, ASA, and AAS.

a. Rewrite the SSS Congruence Theorem, replacing the word "triangles" with "parallelograms." Prove or disprove that your statement about two parallelograms is a theorem.

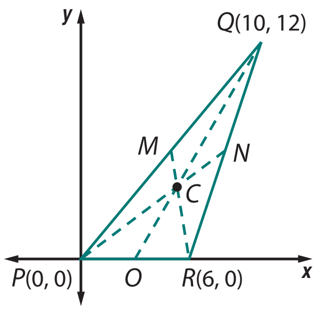
b. Follow the directions in Part a for the SAS Congruence Theorem.

c. Follow the directions in Part a for the ASA Congruence Theorem.

d. Follow the directions in Part a for the AAS Congruence Theorem.

15 Two points determine a line. Explain and provide an illustration that shows why three noncollinear points determine a circle.

16 Find the coordinates of the centroid C of $\triangle PQR$.

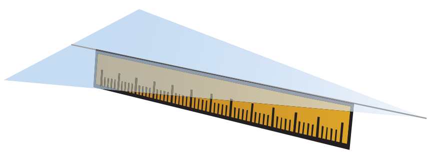


a. For each median, how does the distance from the vertex to point C compare to the length of the median from that vertex?

b. Use interactive geometry software to test if the relationship you found in Part a holds for other triangles. Write a statement summarizing your findings.

17 In Investigation 2, you saw that a cardboard triangle can be balanced on the tip of a pencil placed at the centroid of the triangle.

a. Cut a triangle from poster board or cardboard. Draw a median of the triangle. Can you balance the triangle on the edge of a ruler placed along the median?

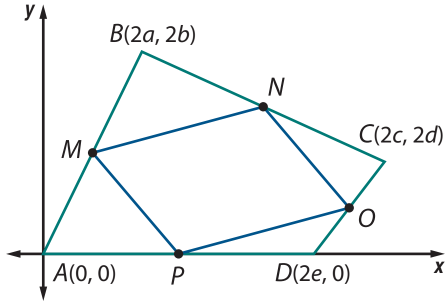


b. Your work in Part a suggests that a median divides a triangle into two triangles of equal area. Write a proof of this statement.

18 In Problem 7 of Investigation 3, you likely used the Midpoint Connector Theorem for Triangles to prove the Midpoint Connector Theorem for Quadrilaterals.

If the midpoints of the sides of a quadrilateral are connected in order, the resulting quadrilateral is a parallelogram.

Consider now how you could prove the statement using coordinate methods.



a. Why was the coordinate system placed so that vertex A is at the origin and side $\overline{AD}$ is on the x-axis?

b. Why were the coordinates of vertices B, C, and D chosen to be $(2a, 2b)$, $(2c, 2d)$, and $(2e, 0)$, respectively?

c. Write a coordinate proof of the Midpoint Connector Theorem for Quadrilaterals.

d. Describe a second strategy for proving this theorem using coordinates.

19 Use interactive geometry software or paper and pencil to explore special cases of the Midpoint Connector Theorem for Quadrilaterals on page 206.

a. Draw a square and name it *ABCD*. Find the midpoints of the sides. Connect them in order to obtain quadrilateral *EFGH*.

i. What special kind of quadrilateral does quadrilateral *EFGH* appear to be?

ii. How would you justify your conjecture?

b. Draw a rectangle and name it *ABCD*. Find the midpoints of the sides. Connect them in order to obtain quadrilateral *EFGH*.

i. What special kind of quadrilateral does quadrilateral *EFGH* appear to be?

ii. How would you justify your conjecture?

c. Draw a rhombus and name it *ABCD*. Find the midpoints of the sides. Connect them in order to obtain quadrilateral *EFGH*.

i. What special kind of quadrilateral does quadrilateral *EFGH* appear to be?

ii. How would you justify your conjecture?

Reflections

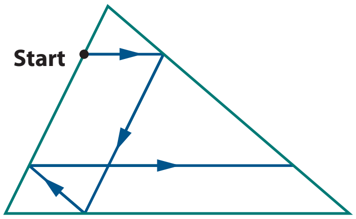
20 How could you generalize the definition of congruence of triangles (page 196) to a definition of *congruence of polygons*?

21 In this unit, you have reasoned with "corresponding angles" in the contexts of similarity and congruence of triangles and in the context of parallel lines cut by a transversal. Draw sketches illustrating the different meanings of "corresponding angles."

22 Given two angles, there is a variety of conditions that would allow you to conclude, without measuring, that the angles are congruent. For example, if you know the angles are vertical angles, then you know they are congruent. What other conditions will allow you to conclude two angles are congruent? List as many as you can.

23 The Midpoint Connector Theorem for Quadrilaterals is an amazing result. It was first proved by a French mathematician, Pierre Varignon (1654–1722). Here is a more recent unexpected "point connector" result.

a. On a copy of the diagram at the right, continue to draw segments parallel to the sides of the triangle. What happens?

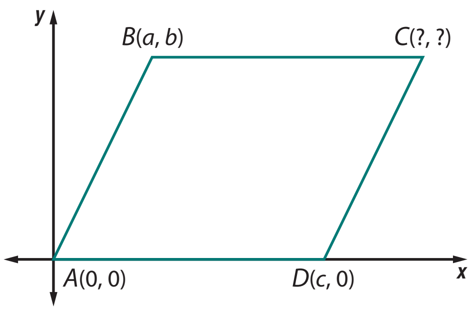


b. What was the total number of segments drawn? Can you explain why this happens?

c. Is there a "start" point for which there will be fewer segments drawn? If so, where is the point located? How many segments will there be?

24 The diagram below can be used to prove properties of parallelograms using coordinates.

a. Determine the coordinates of vertex C.



b. Look back at Problem 3 (pages 205–206) of Investigation 3. From the list, pick two properties of parallelograms that you proved using congruent triangles. For each property, describe how you could also prove the property using coordinate methods.

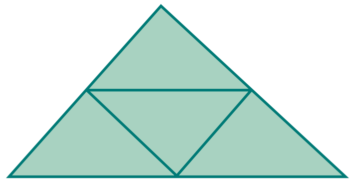
25 In this lesson, you re-examined congruence of triangles as a special case of similarity of triangles. The following statement relates similarity to congruence in terms of a size transformation.

Two figures are similar if one is congruent to a size transformation of the other.

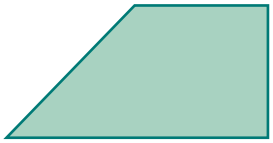
Is this description of similarity consistent with the definition of similarity in Lesson 1? Explain.

Extensions

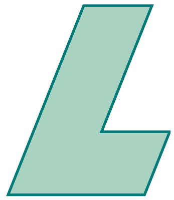
26 *Dissection* of shapes into several congruent shapes is a source for commercial games and interesting geometric problems. The triangle below is dissected into four congruent shapes, each similar to the large triangle. Dissect each of the other two shapes into four shapes similar to the given shape but congruent to each other. Shapes that can be dissected into congruent shapes similar to the original are called **rep-tiles**.



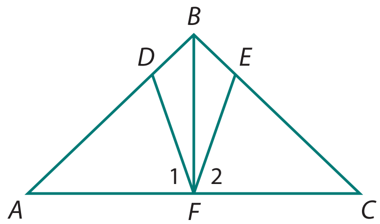
a.



b.



27 In the diagram at the right, $\overline{BF}$ is the perpendicular bisector of $\overline{AC}$ and $\angle 1 \cong \angle 2$. Prove that $\overline{DB} \cong \overline{BE}$.



28 Use interactive geometry software or a compass and straightedge to construct a regular pentagon and a regular hexagon.

a. What appears to be true about the perpendicular bisectors of the sides?

b. Does your observation in Part a also hold for a regular quadrilateral?

c. Make a conjecture about the concurrency of the perpendicular bisectors of the sides of any regular polygon.

d. How could you prove your conjecture using the fact that the perpendicular bisector of a line segment is the set of all points equidistant from the segment's endpoints?

29 Use interactive geometry software or a compass and straightedge to construct a regular pentagon and a regular hexagon.

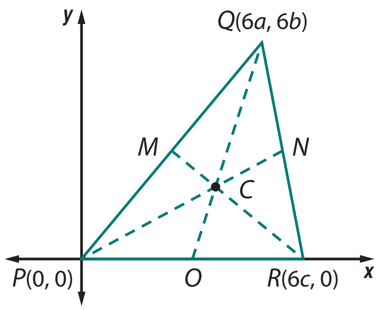
a. What appears to be true about the angle bisectors?

b. Does your observation in Part a also hold for a regular quadrilateral?

c. Make a conjecture about the concurrency of the angle bisectors of any regular polygon.

d. How could you prove your conjecture using the fact that the bisector of an angle is the set of all points equidistant from the sides of the angle?

30 Coordinate methods can be used to prove that the medians of a triangle are concurrent, and that the distance from each vertex to the point of concurrency is $\frac{2}{3}$ the length of the median.

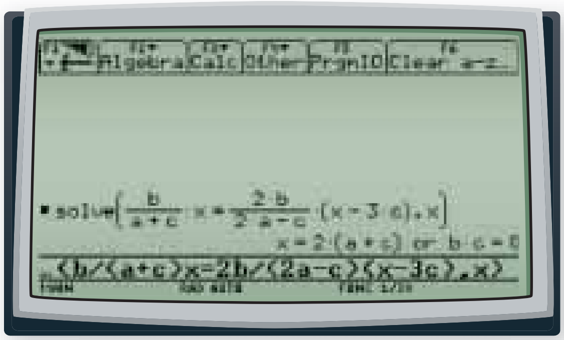


a. Why do you think the coordinates of points P, Q, and R were chosen as shown?

b. To prove that the medians are concurrent, you might proceed as follows.

i. Verify that the equation for $\overleftrightarrow{PN}$ is $y = \frac{b}{a + c}x$. That the equation for $\overleftrightarrow{QO}$ is $y = \frac{2b}{2a - c}(x - 3c)$.

ii. What does the following CAS display tell you?



iii. Verify that the point of intersection of $\overleftrightarrow{PN}$ and $\overleftrightarrow{QO}$ is also on $\overleftrightarrow{MR}$. What can you conclude?

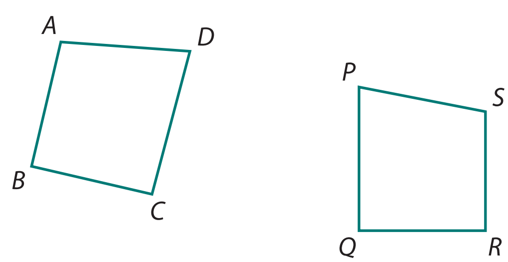
c. Show that the distance from each vertex to the point of concurrency is $\frac{2}{3}$ the length of the median.

31 Look back at Connections Task 14. What conditions that ensure congruence of two parallelograms could be modified with additional side or angle information to provide a test of congruence of two general quadrilaterals? Write an argument justifying your congruence condition for general quadrilaterals.

32 The two quadrilaterals below are not drawn to scale. In Parts a and b, determine if the given information is sufficient to guarantee the two quadrilaterals are congruent. Explain your reasoning. If quadrilateral *ABCD* is not congruent to quadrilateral *PQRS*, what is the least amount of additional information you would need to conclude the two quadrilaterals are congruent?

a. Is *quadrilateral ABCD* $\cong$ *quadrilateral PQRS* if these four conditions are satisfied?

- $\overline{AD} \cong \overline{PS}$
- $\overline{BC} \cong \overline{QR}$
- $\overline{AB}$ is parallel to $\overline{CD}$.
- $\overline{PQ}$ is parallel to $\overline{RS}$.



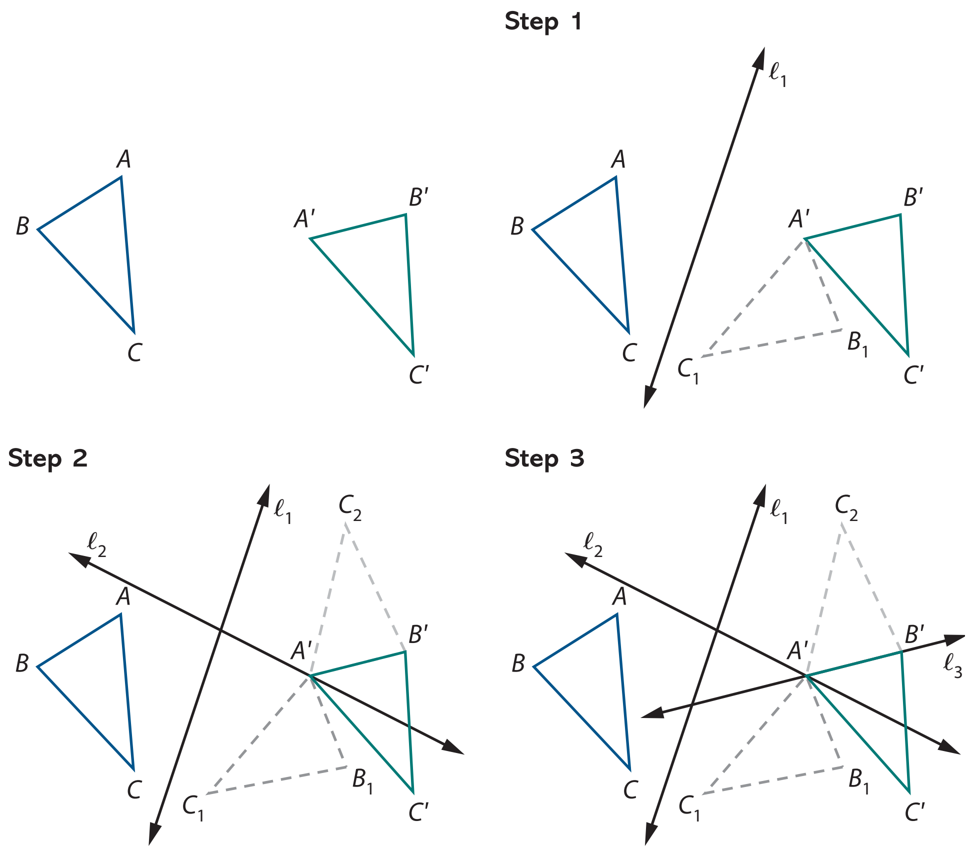
b. Is *quadrilateral ABCD* $\cong$ *quadrilateral PQRS* if these three conditions are satisfied?

- $\angle A \cong \angle P$
- $\angle C \cong \angle R$
- $\overline{AD} \cong \overline{PS}$

33 In Investigation 4, you proved that a triangle and its image under a line reflection or a composition of two (three) line reflections are congruent. In this task, you will justify the following striking result.

If $\triangle ABC \cong \triangle A'B'C'$, then there is a congruence-preserving transformation that maps $\triangle ABC$ onto $\triangle A'B'C'$ and this transformation is the composition of at most three line reflections.

Study the steps below to find the possible lines of reflection for a transformation that will map $\triangle ABC$ onto $\triangle A'B'C'$ by successive reflections across those lines.



a. In Step 1, $\triangle A'B_1C_1$ is the reflection image of $\triangle ABC$ across line ℓ_1.

- How was line ℓ_1 chosen?
- How is $A'B_1$ related to $A'B'$?

b. In Step 2, ℓ_2 was chosen to be the perpendicular bisector of $\overline{B_1B'}$.

- Why was ℓ_2 chosen that way?
- The diagram shows that point A' is on ℓ_2. Why must that be the case?
- How is $A'C_2$ related to $A'C'$?

c. In Step 3, ℓ_3 was chosen to be the perpendicular bisector of $\overline{C_2C'}$.

- Why was ℓ_3 chosen that way?
- The diagram shows that points A' and B' are on ℓ_3. Why must that be the case?

d. Why is A' the reflection image of point A across line ℓ_1, then line ℓ_2, and then line ℓ_3? Why is B' the reflection image of point B under the composition of these three line reflections? Why is C' the reflection image of point C under this composition of line reflections?

e. Suppose in Step 1, the reflection image of point B across ℓ_1 was point B'. How should ℓ_2 in Step 2 be chosen?

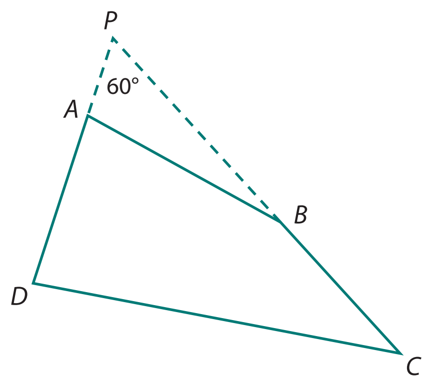
f. Suppose in Step 2, the reflection image of point C_1 was point C'. How should ℓ_3 in Step 3 be chosen?

g. How does your work in Parts a–f justify this fundamental theorem of congruence-preserving transformations?

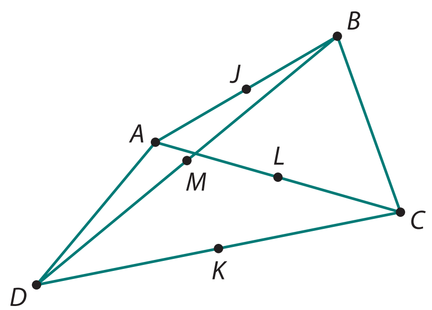
34 An **equilic quadrilateral** is a quadrilateral with a pair of congruent opposite sides that, when extended, meet to form a 60° angle. The other two sides are called *bases*.

a. Quadrilateral $ABCD$ is equilic with bases $\overline{AB}$ and $\overline{CD}$ and $\overline{AD} \cong \overline{BC}$.

- Could base $\overline{AB}$ be parallel to base $\overline{CD}$? Explain your reasoning.
- Could $\overline{AB} \cong \overline{CD}$? Explain.
- Find $m\angle D + m\angle C$.
- Find $m\angle A + m\angle B$.



b. Suppose you have an equilic quadrilateral $ABCD$ with $\overline{AD} \cong \overline{BC}$ and diagonals $\overline{AC}$ and $\overline{BD}$. Points J, K, L, and M are midpoints of bases $\overline{AB}$ and $\overline{CD}$ and diagonals $\overline{AC}$ and $\overline{BD}$, respectively. How are the four points J, K, L, and M related? Prove you are correct.



c. Draw an equilic quadrilateral $ABCD$ with $\overline{AD} \cong \overline{BC}$ and $AB < CD$. Draw an equilateral triangle on base $\overline{AB}$ that shares no points with the interior of quadrilateral $ABCD$. Label the third vertex P. How are points P, C, and D related? Prove your claim.

Review

35 If $1{,}000 < x < 2{,}000$, place the following expressions in order from smallest to largest.

$$\sqrt{x} \qquad \frac{x}{5} \qquad \frac{5}{x} \qquad x^2 \qquad \frac{x+3}{5} \qquad \frac{5}{x+2} \qquad |x|$$

36 For each table of values, determine if the relationship is linear, exponential, or quadratic. Then find a rule that matches the relationship in the table.

a.

x	0	1	2	3	4	5
y	297	99	33	11	$3\frac{2}{3}$	$1\frac{2}{9}$

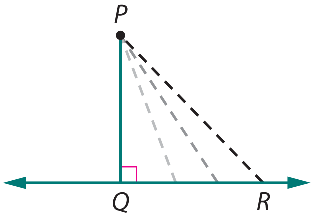
b.

x	0	3	6	9	12	15
y	99	115	131	147	163	179

c.

x	2	3	4	5	6	7
y	18	15.5	13	10.5	8	5.5

37 Why is the length of the perpendicular segment to a line drawn from a point not on a line the shortest distance from the point to the line?



38 Find the length of the line segment that connects each pair of points. Express your answer in simplest radical form.

a. (3, 6) and (11, 10)

b. (5, −4) and (0, −1)

c. (a, b) and $(a, 0)$

39 Consider $\triangle XYZ$ with $m\angle Y = 90°$ and $\sin X = \frac{3}{5}$.

a. Find $m\angle X$.

b. Sketch two different triangles that each meet the given conditions. Identify the lengths of all sides of your triangles.

c. Are the two triangles you drew in Part b similar? Explain your reasoning.

40 Rewrite each expression in a simpler equivalent form.

a. $-\frac{2}{5}(14x + 12) + \frac{1}{5}(3x - 6)$

b. $\frac{8x + 10}{2} + 6x - 15$

c. $x(10 - x) - 3(x + 4) + 12$

41 The suggested serving size for a certain cereal is $1\frac{1}{4}$ cups.

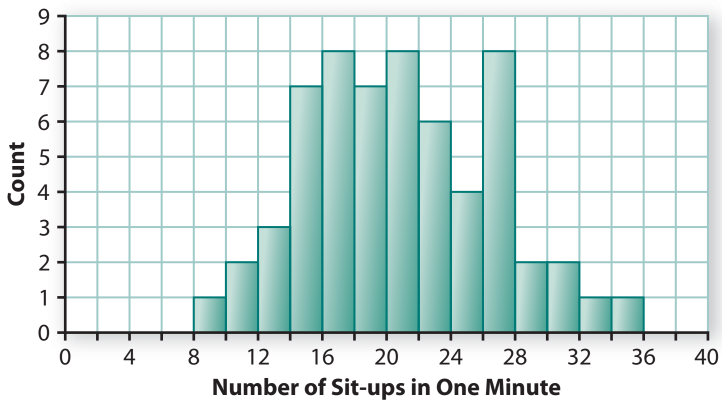
a. How much cereal would you need if you wanted to have 15 servings?

b. Georgia has 25 cups of cereal. How many servings does she have?

c. A box of cereal weighs 10 ounces and contains 9 servings. What does 1 cup of cereal weigh?

d. Write a symbolic rule that describes the relationship between the weight of the cereal and the number of servings.

42 The histogram below represents the number of sit-ups that the 60 eleventh-grade students at Eisenhower High School were able to complete in one minute.



a. Jeremy did 29 sit-ups. What is his percentile ranking within this group of eleventh graders?

b. Stephanie was only at the 40th percentile. How many sit-ups was Stephanie able to do?

43 Graph the solution set to each system of inequalities. On your graphs, identify the coordinates of the points of intersections of the graphs.

a. $4x + 8y < 40$
$x - 2y > 8$

b. $y \geq -3$
$y \leq x^2 - 5$

44 Suppose a data set x_1, x_2, x_3, x_4, x_5, has mean $\overline{x}$ and standard deviation s.

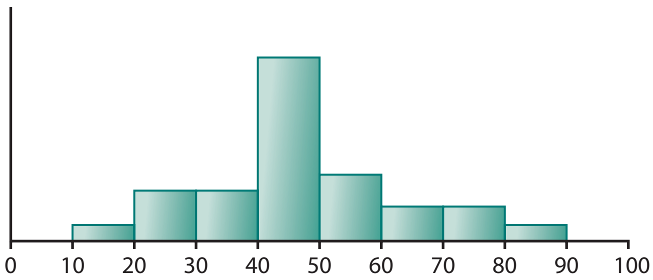
a. If a constant c is added to each of the five values, what is the mean of the transformed values? What is the standard deviation of the transformed values?

b. If each of the five values is multiplied by a positive constant d, what is the mean of the transformed values? What is the standard deviation of the transformed values?

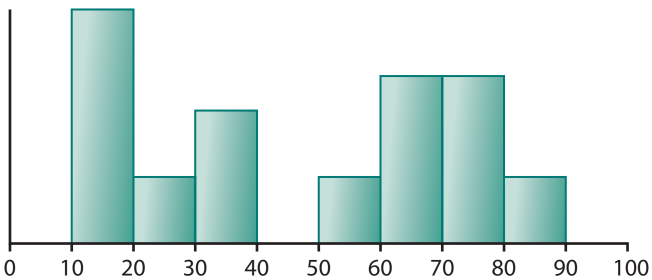
c. How would the data transformation in Part a affect the median? The interquartile range?

d. How would the data transformation in Part b affect the median? The interquartile range?

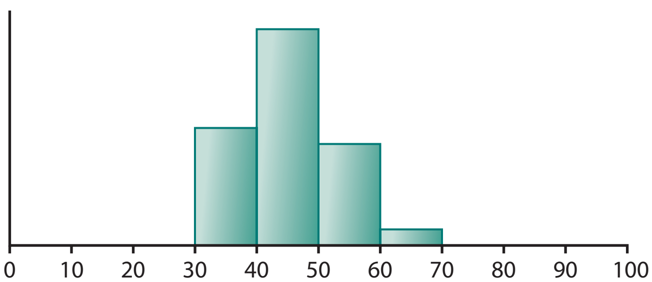
45 Match each of the following histograms of test scores in three classes, I, II, and III, to the best description of class performance.



Test Scores for Class I



Test Scores for Class II



Test Scores for Class III

a. The mean of the test scores is 46, and the standard deviation is 26.

b. The mean of the test scores is 46, and the standard deviation is 8.

c. The mean of the test scores is 46, and the standard deviation is 16.

LESSON 3

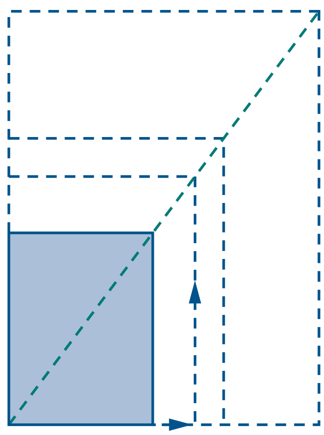
Looking Back

In this unit, you re-examined similarity and congruence as related ideas and as tools for solving problems. In Lesson 1, you used the Law of Cosines and the Law of Sines to deduce sufficient conditions on side lengths or angle measures to ensure similarity of triangles. You used similarity to explain the design of mechanisms, calculate inaccessible heights and distances, and prove important geometric relationships and properties of size transformations.

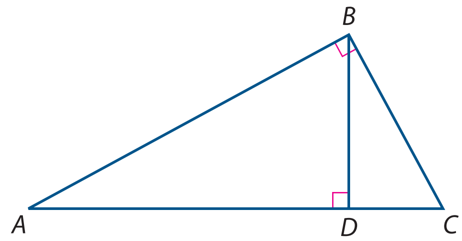
In Lesson 2, you used similarity conditions for triangles to justify sufficient conditions for congruence of triangles that you discovered and used without proof in your previous work. You used congruence of triangles to locate special centers of triangles and to investigate and prove properties and conditions for congruence of (special) parallelograms. You also re-examined and proved properties of congruence-preserving transformations and then used those transformations to solve problems.

The tasks in this final lesson will help you review and organize your knowledge of key ideas and strategies for reasoning with similarity and congruence.

1. When a rectangular photograph is to be enlarged to fit the width of a given column of type in a newspaper or magazine, the new height can be found by extending the width to the new measure and drawing a perpendicular as shown. The point at which the perpendicular meets the extended diagonal determines the height of the enlargement. Explain as precisely as you can why this method works.



2 In the diagram below, $\triangle ABC$ is a right triangle and $\overline{BD}$ is an altitude to side $\overline{AC}$.

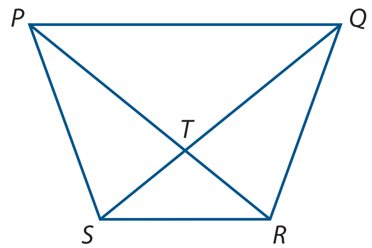


a. Prove that $(AB)^2 = (AC)(AD)$.

b. Find a similar expression for $(BC)^2$.

c. Use your work in Parts a and b to provide another proof of the Pythagorean Theorem.

3 In the diagram below, $\overline{PR}$ and $\overline{QS}$ are diagonals of quadrilateral $PQRS$, and interest at point T. $\overline{PS} \cong \overline{QR}$, $\overline{PT} \cong \overline{QT}$, and $\overline{ST} \cong \overline{RT}$.



a. Identify and name eight triangles in the diagram.

b. Which pairs of the eight triangles seem to be congruent? Name each pair.

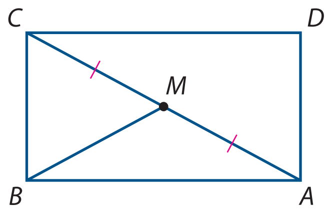
c. Using the given information, prove or disprove that each identified pair of triangles are congruent.

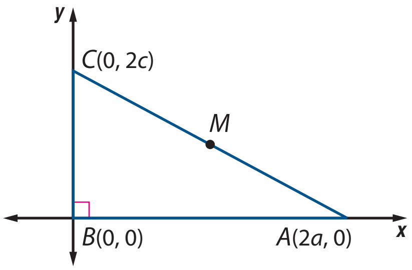
d. Which of the remaining pairs of triangles are similar? Provide an argument to support your answer.

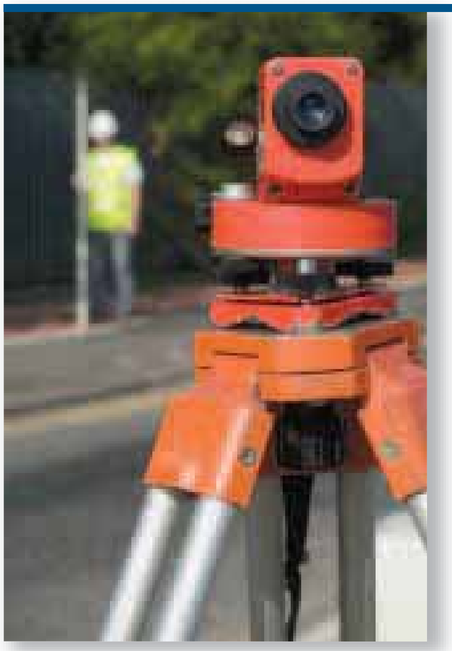
e. What can you conclude about $\overline{PQ}$ and $\overline{SR}$ when you know the given information? Write a proof for your claim.

4 Civil engineers, urban planners, and design engineers are frequently confronted with "traffic center" problems. In mathematical terms, these problems often involve locating a point M that is equidistant from three or more given points or for which the sum of the distances from M to the given points is as small as possible. In the first two parts of this task, you are to write and analyze a synthetic proof and a coordinate proof of the following theorem:

The midpoint of the hypotenuse of a right triangle is equidistant from its vertices.





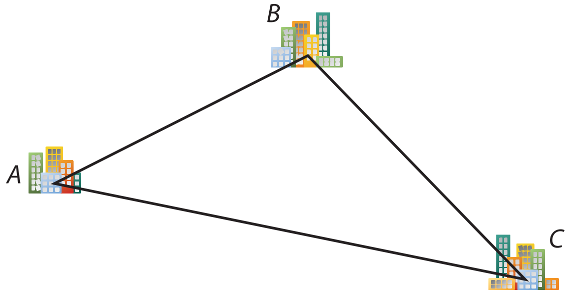


a. In the diagram on the left, $\triangle ABC$ is composed of two consecutive sides and a diagonal of rectangle $ABCD$. Using that diagram, write a synthetic proof that if M is the midpoint of $\overline{AC}$, then $MA = MB = MC$.

b. Using the diagram on the right, write an analytic proof that if M is the midpoint of $\overline{AC}$, then $MA = MB = MC$.

c. For the proof in Part b, why were the coordinates of points A and C assigned $(2a, 0)$ and $(0, 2c)$ rather than simply $(a, 0)$ and $(0, c)$, respectively?

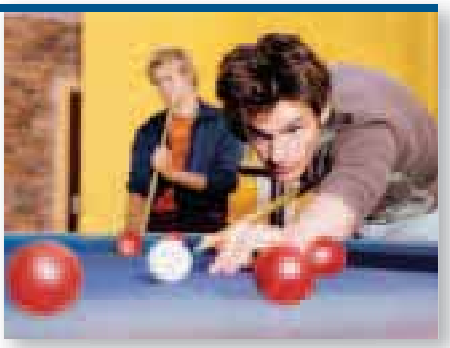
5 Suppose cities A, B, and C are connected by highways $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ as shown.

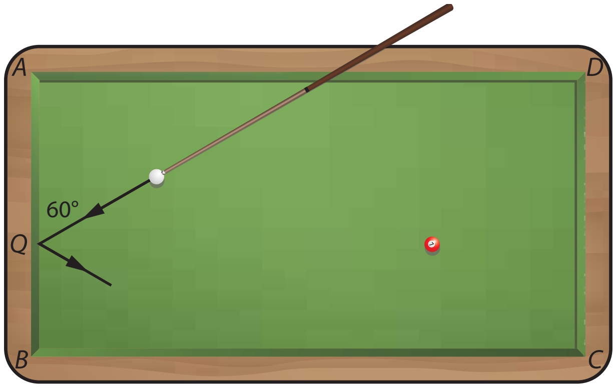


a. On a copy of $\triangle ABC$, find the location for a proposed new airport that would be the same distance from each of the existing highways.

b. Explain why your proposed location meets the constraints.

6 On a copy of the billiards table, sketch the path of the cue ball as it rebounds from $\overline{AB}$, from $\overline{BC}$, from $\overline{CD}$, from $\overline{AD}$, and from $\overline{AB}$ again. (This assumes that the ball has enough speed to travel the whole path, of course!) Label the points where the ball strikes the cushions R, S, T, and U, respectively.





a. At what angle does the ball leave $\overline{AB}$ the second time? Explain your reasoning.

b. What appears to be true about $\overline{QR}$ and $\overline{ST}$? Write a proof of your conjecture.

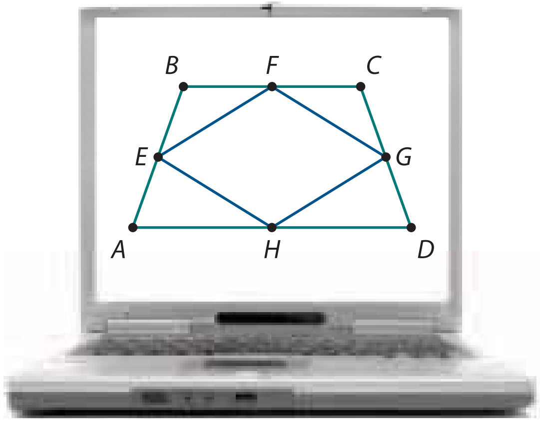
c. Suppose you wanted to bank the cue ball off $\overline{BC}$ rather than $\overline{AB}$ in order to strike the red ball. At what point on the cushion $\overline{BC}$ would you aim?

- On a copy of the billiards table, draw the path of the cue ball.
- Prove that if the ball has no spin and sufficient speed, it will follow the drawn path and strike the red ball.

d. Is it possible, for each side of the table, to bank the cue ball off that side and strike the red ball? Explain.

e. If you were to try to hit the red ball by banking the cue ball off two sides, which pair of sides would you use? Why? Draw the path of the cue ball. Justify that it will hit the red ball.

7 The display below suggests that if the midpoints of the sides of an isosceles trapezoid are connected in order, a special parallelogram is formed.



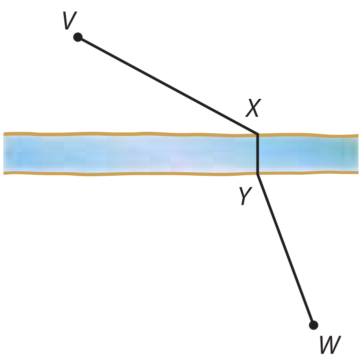
a. Make and test conjectures about the type of parallelogram that is formed.

b. Prove your conjecture using synthetic methods. Using coordinate methods.

c. Which proof method in Part b seems to be the better approach for this problem? Why?

8 In July 2006, a tsunami, triggered by an undersea earthquake, struck the coasts of India and Indonesia, destroying villages. Tidal waves damaged bridges. A new bridge connecting villages V and W positioned as shown below is to be constructed across the river.

a. If river conditions permit, why should a bridge $\overline{XY}$ be constructed perpendicular to the banks of the river?



b. The diagram shows one possible location of the bridge. Write an expression for the length of the path from village V to village W.

c. On a copy of this diagram, find the image V' of point V under the translation that maps point X onto point Y. How could you use point V' to find a location for the bridge so that the path from point V to the bridge and then to point W is of minimum length? Why would that make sense?

d. Explain why your location of the bridge provides a shorter path than the path in the diagram. Why is it shorter than the path for any other possible location of the bridge?

Summarize the Mathematics

In this unit, you have extended your understanding of similarity and congruence and sharpened your skills in mathematical reasoning.

a What conditions are sufficient to guarantee that two triangles are similar? Are congruent?

b How are the ideas of congruence and similarity related?

c What is the Midpoint Connector Theorem for Triangles? For Quadrilaterals? What are strategies that can be used to remember and prove each theorem?

d A circle has one center. But a triangle has more than one center. Identify and describe how to locate each center studied in this unit. Then describe an application for each.

e Identify at least two conditions on a quadrilateral that guarantees that it is a parallelogram. A rhombus. A rectangle.

f Suppose in a diagram, $\triangle X'Y'Z'$ is the image of $\triangle XYZ$ under the indicated transformation.

- **i.** Size transformation of magnitude k: How could you locate the center of the size transformation and determine the magnitude k?
- **ii.** Line reflection: How could you determine the line of reflection?
- **iii.** Rotation: How could you determine the center of the rotation and the directed angle of rotation?
- **iv.** Translation: How could you determine lines ℓ_1 and ℓ_2 so that the translation can be expressed as a composition of reflections across those lines? How are the magnitude and direction of the translation related to lines ℓ_1 and ℓ_2?

g How are similarity and congruence related to size transformations and congruence-preserving transformations?

Be prepared to explain your ideas and strategies to the class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.