

UNIT 8

PATTERNS IN CHANCE

Many events in life occur by chance. They cannot be predicted with certainty. For example, you cannot predict whether the next baby born will be a boy or a girl. However, in the long run, you can predict that about half of the babies born will be boys and half will be girls. Probability is the study of events that occur (or don't occur) at random from one observation to the next but that occur a fixed proportion of the time in the long run. In this unit, you will study two methods for solving problems in probability: calculating theoretical probabilities using mathematical formulas or using geometric models and estimating probabilities using simulation.

Key ideas for solving problems involving chance will be developed through your work in two lessons.



Lessons

1 *Calculating Probabilities*

Construct probability distributions from sample spaces of equally likely outcomes. Use the Addition Rule to solve problems involving chance.

2 *Modeling Chance Situations*

Design simulations using a table of random digits or random number generators to estimate answers to probability questions. Use geometric models to solve probability problems.

LESSON 1



Calculating Probabilities

Backgammon is the oldest game in recorded history. It originated before 3000 B.C. in the Middle East. Backgammon is a two-person board game that is popular among people of all ages. When it is your turn, you roll a pair of dice and move your checkers ahead on the board according to what the dice show. The object is to be the first to move your checkers around and then off of the board. It's generally a good thing to get doubles because then you get to use what the dice show twice instead of just once.

Think About This Situation

Suppose you and a friend are playing a game of backgammon.

- a** Which probability should be larger?
- the probability of rolling doubles on your first turn
 - the probability of rolling doubles on your first turn or on your second turn
- Explain your thinking.
- b** What do you think is the probability of rolling doubles on your first turn? Explain your reasoning.
- c** What assumptions are you making about the dice in finding the probability in Part b?
- d** Suppose you rolled doubles on each of your first three turns. Your friend did not roll doubles on any of his first three turns. Who has the better chance of rolling doubles on their next turn? Explain.

In this lesson, you will learn to construct probability distributions from sample spaces of *equally likely outcomes* and use them to solve problems involving chance.

Investigation 1 Probability Distributions

Because of the symmetry in a fair die—each side is equally likely to end up on top when the die is rolled—it is easy to find the probabilities of various outcomes. As you work on the problems in this investigation, look for answers to this question:

How can you find and organize the probabilities associated with random events like the roll of two dice?

- 1** Suppose a red die and a green die are rolled at the same time.
- What does the entry “3, 2” in the chart mean?
 - Complete a copy of the chart at the right, showing all possible outcomes of a single roll of two dice.
 - How many possible outcomes are there?
 - What is the probability of rolling a (1, 2), that is, a 1 on the red die and a 2 on the green die? What is the probability of rolling a (2, 1)? A (4, 4)?
 - Would the chart be any different if both dice had been the same color?



Rolling Two Dice

		Number on Green Die					
		1	2	3	4	5	6
Number on Red Die	1	1, 1					
	2						
	3		3, 2				
	4					4, 5	
	5						
	6						

- 2 The chart you completed in Problem 1 is called a sample space for the situation of rolling two dice. A **sample space** is a list of all possible outcomes. For fair dice, all 36 outcomes in the sample space are equally likely to occur. **Equally likely** means that each outcome has the same probability. When outcomes are equally likely, the probability of an event is given by

$$P(\text{event}) = \frac{\text{number of successful outcomes}}{\text{number of possible outcomes}}$$

If two dice are rolled, what is the probability of getting

- doubles?
 - a sum of 7?
 - a sum of 11?
 - a 2 on at least one die or a sum of 2?
 - doubles and a sum of 8?
 - doubles or a sum of 8?
- 3 Suppose two dice are rolled.
- What is the probability that the sum is no more than 9?
 - What is the probability that the sum is at least 9?
 - What is the probability that the sum is 2 or 3? Is greater than 3? Is at least 3? Is less than 3?
- 4 A **probability distribution** is a description of all possible numerical outcomes of a random situation, along with the probability that each occurs. A probability distribution differs from a sample space in that all of the outcomes must be a single number and the probabilities must be specified. For example, the probability distribution table below shows all possible sums that you could get from the roll of two dice.

**Probability Distribution
for the Sum of Two Dice**

Sum	Probability
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

- a. Complete a copy of this probability distribution by filling in the probabilities.
- b. What is the sum of all of the probabilities?
- c. How could you use your probability distribution table to find each of the probabilities in Problem 3?

5 Other probability distributions can be made from the sample space in Problem 1 for the roll of two dice. Suppose that you roll two dice and record the larger of the two numbers. (If the numbers are the same, record that number.)

- a. Use your sample space from Problem 1 to help you complete a probability distribution table for this situation.
- b. What is the probability that the larger of the two numbers is 3? Is 2 or 3? Is 3 or less? Is more than 3?

6 Now suppose you roll two dice and record the absolute value of the difference of the two numbers.

- a. Use your sample space from Problem 1 to help you complete a probability distribution table for this situation.
- b. What is the probability that the absolute value of the difference is 3? Is 2 or 3? Is at least 2? Is no more than 2?

7 If you flip a coin, {heads, tails} is a sample space, but not a probability distribution. However, you can make a probability distribution by recording the number of heads as your outcome, as shown in the table below. Fill in the two missing probabilities.

**Probability Distribution
for the Number of Heads**

Number of Heads	Probability
0	
1	



- 8** Now suppose that you flip a coin twice.
- a. Complete a chart that shows the sample space of all possible outcomes. It should look like the chart for rolling two dice except that only heads and tails are possible for each coin rather than the six numbers that are possible for each die.
 - b. Make a probability distribution table that gives the probability of getting 0, 1, and 2 heads.
 - c. What is the probability that you get exactly one head if you flip a coin twice? What is the probability that you get at least one head?

Summarize the Mathematics

In this investigation, you learned how to construct probability distributions from the sample space of equally likely outcomes from the roll of two dice and from the flip(s) of a coin.

- a What is the difference between a sample space and a probability distribution?
- b How would you make a probability distribution table for the product of the numbers from the roll of two dice?
- c Why is it that the outcomes in the sample space for rolling two dice are equally likely?

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

Suppose that you flip a coin three times.

- a. List the sample space of all 8 possible outcomes. For example, one outcome is heads, tails, tails (HTT).
- b. Are the outcomes in your sample space equally likely? Explain.
- c. Make a probability distribution table for the number of heads.
- d. What is the probability that you will get exactly 2 heads? At most 2 heads?

Investigation 2 The Addition Rule

In the previous investigation, you constructed the probability distribution for the sum of two dice. You discovered that to find the probability that the sum is 2 or 3, you could *add* the probability that the sum is 2 to the probability that the sum is 3, $\frac{2}{36} + \frac{3}{36} = \frac{5}{36}$. As you work on the following problems, look for an answer to this question:

Under what conditions can you add individual probabilities to find the probability that a related event happens?

- 1 Some people have shoes of many different colors, while others prefer one color and so have all their shoes in just that color. As a class, complete a copy of the following two tables on the color of your shoes.

In the first table, record the number of students in your class that today are wearing each shoe color. (If a pair of shoes is more than one color, select the color that takes up the largest area on the shoes.)

Color of Shoes You Are Wearing Today	Number of Students
Blue	
Black	
White	
Brown, Beige, or Tan	
Red	
Other	

Now complete the second table by recording the number of students in your class who own a pair of shoes of that color. For example, a student who has all shoes in the colors blue or black would identify themselves for only those two colors.

Color of Shoes You Own	Number of Students
Blue	
Black	
White	
Brown, Beige, or Tan	
Red	
Other	

In mathematics, the word “or” means “one or the other or both.” So, the event that *a student owns white shoes or owns black shoes* includes all of the following outcomes:

- The student owns white shoes but doesn’t own black shoes.
 - The student owns black shoes but doesn’t own white shoes.
 - The student owns both white and black shoes.
- a.** Which question below can you answer using just the data in your tables? Answer that question.
- I** What is the probability that a randomly selected student from your class is wearing shoes today that are black or wearing shoes that are white?
- II** What is the probability that a randomly selected student from your class owns shoes that are black or owns shoes that are white?
- b.** Why can’t the other question in Part a be answered using just the information in the tables?
- c.** As a class, collect information that can be used to answer the other question.

- 2 The table below gives the percentage of high school sophomores who say they engage in various activities at least once a week.

Weekly Activities of High School Sophomores

Activity	Percentage of Sophomores
Use personal computer at home	71.2
Drive or ride around	56.7
Work on hobbies	41.8
Take sports lessons	22.6
Take class in music, art, language	19.5
Perform community service	10.6

Source: National Center for Education Statistics. *Digest of Education Statistics 2004*, Table 138. Washington, DC: nces.ed.gov/programs/digest/d04/tables/dt04_138.asp

Use the data in the table to help answer, if possible, each of the following questions. If a question cannot be answered, explain why not.

- What is the probability that a randomly selected sophomore takes sports lessons at least once a week?
- What is the probability that a randomly selected sophomore works on hobbies at least once a week?
- What is the probability that a randomly selected sophomore takes sports lessons at least once a week or works on hobbies at least once a week?
- What is the probability that a randomly selected sophomore works on hobbies at least once a week or uses a personal computer at home at least once a week?



- 3 You couldn't answer the "or" questions in Problem 2 by adding the numbers in the table. However, you could answer the "or" questions in Problems 3, 4, 5, and 6 of the previous investigation by adding individual probabilities in the tables. What characteristic of a table makes it possible to add the probabilities to answer an "or" question?

- 4 The Minnesota Student Survey asks teens questions about school, activities, and health. Ninth-graders were asked, “How many students in your school are friendly?” The numbers of boys and girls who gave each answer are shown in the table below.

How Many Students in Your School Are Friendly?

Answer	Boys	Girls	Total
All	480	303	783
Most	13,199	14,169	27,368
Some	7,920	8,874	16,794
A few	1,920	1,815	3,735
None	480	50	530
Total	23,999	25,211	49,210

Source: 2004 Minnesota Student Survey, Table 4,
www.mnschoolhealth.com/resources.html?ac=data

Suppose you pick one of these students at random.

- Find the probability that the student said that all students are friendly.
- Find the probability that the student said that most students are friendly.
- Find the probability that the student is a girl.
- Find the probability that the student is a girl and said that all students are friendly.
- Think about how you would find the probability that the student said that all students are friendly or said that most students are friendly. Can you find the answer to this question using your probabilities from just Parts a and b? If so, show how. If not, why not?
- Think about the probability that the student is a girl or said that all students are friendly.
 - Can you find the answer to this question using just your probabilities from Parts a and c? If so, show how. If not, why not?
 - Can you find the answer if you also can use your probability in Part d? If so, show how. If not, why not?

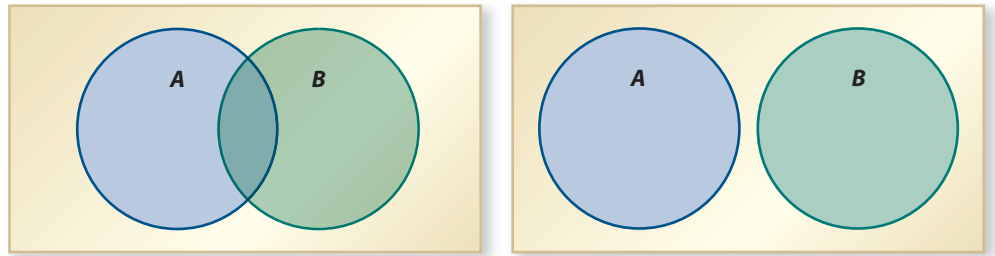
- 5 Two events are said to be **mutually exclusive** (or **disjoint**) if it is impossible for both of them to occur on the same outcome. Which of the following pairs of events are mutually exclusive?
- You roll a sum of 7 with a pair of dice; you get doubles on the same roll.
 - You roll a sum of 8 with a pair of dice; you get doubles on the same roll.
 - Isaac wears white shoes today to class; Isaac wears black shoes today to class.



- d. Yen owns white shoes; Yen owns black shoes.
- e. Silvia, who was one of the students in the survey described in Problem 3, works on hobbies; Silvia plays a sport.
- f. Pat, who was one of the students in the survey described in Problem 3, is a boy; Pat said most students in his school are friendly.
- g. Bernardo, who was one of the students in the survey described in Problem 3, said all students are friendly; Bernardo said most students are friendly.

6 Suppose two events A and B are mutually exclusive.

- a. Which of the *Venn diagrams* below better represents this situation?



- b. What does the fact that A and B are mutually exclusive mean about $P(A \text{ and } B)$ —the probability that A and B both happen on the same outcome?
- c. When A and B are mutually exclusive, how can you find the probability that A happens or B happens (or both happen)?
- d. Write a symbolic rule for computing the probability that A happens or B happens, denoted $P(A \text{ or } B)$, when A and B are mutually exclusive. This rule is called the **Addition Rule for Mutually Exclusive Events**.

7 Suppose two events A and B are *not* mutually exclusive.

- a. Which diagram in Problem 6 better represents this situation?
- b. What does the fact that A and B are not mutually exclusive mean about $P(A \text{ and } B)$? Where is this probability represented on the Venn diagram you chose?
- c. Review your work in Problems 1 and 4 and with the Venn diagram. Describe how you can modify your rule from Problem 6, Part d to compute $P(A \text{ or } B)$ when A and B are not mutually exclusive.
- d. Write a symbolic rule for computing $P(A \text{ or } B)$. This rule is called the **Addition Rule**.

8 Test your rules on the following problems about rolling a pair of dice.

- a. Find the probability that you get doubles or a sum of 5.
- b. Find the probability that you get doubles or a sum of 2.
- c. Find the probability that the absolute value of the difference is 3 or you get a sum of 5.
- d. Find the probability that the absolute value of the difference is 2 or you get a sum of 11.

Summarize the Mathematics

In this investigation, you learned how to compute the probability that event A happens or event B happens on the same outcome.

- a** Give an example of two mutually exclusive events different from those in the investigation.
- b** How do you find the probability that event A happens or event B happens if events A and B are mutually exclusive?
- c** How must you modify the rule in Part b if the two events are not mutually exclusive?

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

Use what you have learned about mutually exclusive events and the Addition Rule to complete the following tasks.

- a.** Which of the following pairs of events are mutually exclusive? Explain your reasoning.
 - i.** rolling a pair of dice: getting a sum of 6; getting one die with a 6 on it
 - ii.** flipping a coin 7 times: getting exactly 3 heads; getting exactly 5 heads
 - iii.** flipping a coin 7 times: getting at least 3 heads; getting at least 5 heads
- b.** Use the appropriate form of the Addition Rule to find the probability of rolling a pair of dice and
 - i.** getting a sum of 6 or getting one die with a 6 on it.
 - ii.** getting a sum of 6 or getting doubles.
- c.** Janet, a 50% free-throw shooter, finds herself in a two-shot foul situation. She needs to make at least one of the shots.
 - i.** List a sample space of all possible outcomes. Are the outcomes equally likely?
 - ii.** Find the probability that she will make the first shot or the second shot.
 - iii.** What assumptions did you make about her shooting?

On Your Own

Applications

- 1 The game of Parcheesi is based on the Indian game pachisi.



In Parcheesi, two dice are rolled on each turn. A player cannot start a pawn until he or she rolls a five. The five may be on one die, or the five may be the sum of both dice. What is the probability a player can start a pawn on the first roll of the dice?

- 2 Suppose you roll a die and then roll it again. The die has the shape of a regular tetrahedron and the numbers 1, 2, 3, and 4 on it.
- a. Make a chart that shows the sample space of all possible outcomes.
- b. How many possible outcomes are there? Are they equally likely?
- c. Make a probability distribution table for the difference of the two rolls (*first die* – *second die*).
- d. What difference are you most likely to get?
- e. What is the probability that the difference is at most 2?
- 3 Suppose that you roll a tetrahedral die and a six-sided die at the same time.
- a. Make a chart that shows the sample space of all possible outcomes.
- b. How many possible outcomes are there? Are they equally likely?
- c. Make a table for the probability distribution of the sum of the two dice.
- d. What sum are you most likely to get?
- e. What is the probability that the sum is at most 3?



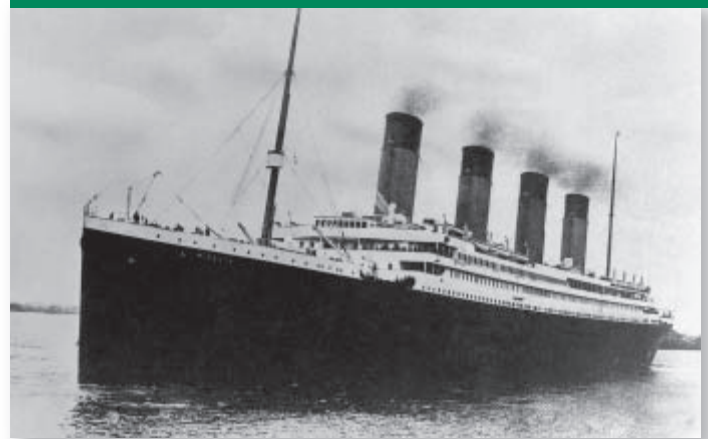
- 4** Use your chart from Part a of Applications Task 2 and the appropriate form of the Addition Rule to answer these questions about a roll of two tetrahedral dice.
- What is the probability that you get a difference of 3 or you get a 2 on the first die?
 - What is the probability you get a difference of 2 or you get doubles?
 - What is the probability you get a difference of 0 or you get doubles?
 - What is the probability you get a difference of 0 or a sum of 6?
- 5** The Titanic was a British luxury ship that sank on its first voyage in 1912. It was en route from Southampton, England, to New York City. The table below gives some information about the passengers on the Titanic.

Passengers Aboard the Titanic

	Men	Women and Children	Total
Survived	138	354	492
Died	678	154	832
Total	816	508	1,324

Source: <http://www.titanicinquiry.org/USInq/USReport/AmlnqRep03.html#a8>

- Suppose a passenger is selected at random. Use the table above to find the probability of each of the following events.
 - The passenger is a man.
 - The passenger survived.
 - The passenger is a man and survived.
- Now use your results from Part a and the appropriate form of the Addition Rule to find the probability that a randomly selected passenger is a man or a survivor. Check your answer by adding the appropriate entries in the table.
- Suppose a passenger is selected at random. Find the probability of each of the following events.
 - The passenger is a woman/child.
 - The passenger died.
 - The passenger is a woman/child and died.
 - The passenger is a woman/child or died.



- 6 In almost all states, it is illegal to drive with a blood alcohol concentration (BAC) of 0.08 grams per deciliter (g/dL) or greater. The table below gives information about the drivers involved in a crash in which someone died. The age of the driver is given in the left column. The BAC of the driver is given across the top row.

Drivers Involved in Fatal Crashes

Age of Driver (in years)	Number with BAC Lower than 0.08 g/dL	Number with BAC 0.08 g/dL or Higher	Total Number of Drivers
16–20	6,395	1,314	7,709
21–24	4,313	2,069	6,382
25–34	8,171	3,008	11,179
35–44	8,217	2,465	10,682
45–54	7,418	1,684	9,102
55–64	4,882	691	5,573
65–74	2,835	222	3,057
75+	2,999	143	3,142
Total	25,230	11,596	56,826

Source: *Traffic Safety Facts, 2004 Data*, National Center for Statistics and Analysis, U.S.

Suppose that you select a driver at random from these 56,826 drivers involved in fatal crashes.

- Estimate the probability that the driver was age 16–20.
- Estimate the probability that the driver was age 21–24.
- Estimate the probability that the driver had a BAC of 0.08 or greater.
- Estimate the probability that the driver was age 16–20 or was age 21–24.
- Can you find the answer to Part d using just your probabilities from Parts a and b? Why or why not?
- Estimate the probability that the driver was age 16–20 or had a BAC of 0.08 or greater.
- Can you find the answer to Part f just by adding the two probabilities from Parts a and c? Why or why not?

Connections

- 7 Make a histogram of the information in the probability distribution table that you created for the sum of two dice in Problem 4 (page 534) of Investigation 1. Probability will go on the y -axis.
- What is the shape of this distribution?
 - What is its mean?

- c. Estimate the standard deviation using the approximation that about two-thirds of the probability should be within one standard deviation of the mean. (While this distribution isn't normal, this approximation works relatively well.)

- 8 Graph the points that represent the information in the probability distribution table that you created for the sum of two dice in Problem 4 (page 534) of Investigation 1. For example, the first point would be $(2, \frac{1}{36})$ and the second point would be $(3, \frac{2}{36})$.

a. Write a linear equation with a graph going through the points for $x = 2, 3, 4, 5, 6, 7$.

b. Write a second linear equation with a graph going through the points for $x = 7, 8, 9, 10, 11, 12$.

c. What are the slopes of these two lines?

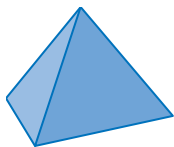
- 9 Graph the points that represent the information in the probability distribution table that you created for the difference of two tetrahedral dice in Applications Task 2. For example, the first point would be $(-3, \frac{1}{16})$ and the second point would be $(-2, \frac{2}{16})$.

a. Write a linear equation with a graph going through the points for $x = -3, -2, -1, 0$.

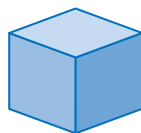
b. Write a second linear equation with a graph going through the points for $x = 0, 1, 2, 3$.

c. What are the slopes of these two lines?

- 10 Recall that there are five regular polyhedra: tetrahedron (4 faces), hexahedron or cube (6 faces), octahedron (8 faces), dodecahedron (12 faces), and icosahedron (20 faces).



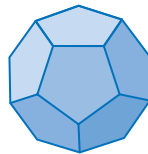
Tetrahedron



Hexahedron



Octahedron



Dodecahedron



Icosahedron

Find or imagine pairs of dice in the shape of these polyhedra.

The numbers on the faces of the tetrahedron are 1, 2, 3, and 4. The hexahedron (regular die) has the numbers 1, 2, 3, 4, 5, and 6. The remaining three pairs of dice have the numbers 1 to 8, 1 to 12, and 1 to 20 on their faces, respectively.

a. Make a sample space chart like that in Investigation 1, Problem 1 (page 533) for the octahedral dice.

b. On which of the five pairs of dice is the probability of getting doubles the greatest? Explain why this is the case.

- c. If the number of faces on a pair of regular polyhedral dice is n , what is the probability (in terms of n) of rolling doubles with that pair of dice?
- d. For each type of dice, what is the mean of the probability distribution of the sum?

- 11 On Problem 6 on page 540, you saw how to draw Venn diagrams to represent the situation where events A and B are mutually exclusive and the situation where events A and B are *not* mutually exclusive. Now, think about three events, A , B , and C .
- a. Draw a Venn diagram that represents the situation where A and B are mutually exclusive, A and C are mutually exclusive, and B and C are mutually exclusive.
 - b. Draw a Venn diagram that represents the situation where A and B are mutually exclusive, A and C are mutually exclusive, but B and C are not mutually exclusive.
 - c. Draw a Venn diagram that represents the situation where A and B are not mutually exclusive, A and C are not mutually exclusive, and B and C are not mutually exclusive.

Reflections

- 12 Some chance situations have exactly two outcomes.
- a. Is it true that the two outcomes must be equally likely? Explain.
 - b. Spin a coin by holding it on edge and flicking it with your finger. Do this 30 times and record the results. Does it appear that heads and tails are equally likely? How could you decide for sure?

- 13 For each phrase below, select the corresponding symbolic phrase:

$$x \geq 9 \qquad x > 9 \qquad x \leq 9 \qquad x < 9$$

- | | |
|--------------------------|--------------------------|
| a. x is no more than 9 | b. x is less than 9 |
| c. x is 9 or greater | d. x is no less than 9 |
| e. x is at most 9 | f. x is at least 9 |
| g. x is greater than 9 | h. x is a maximum of 9 |
| i. x is 9 or more | |

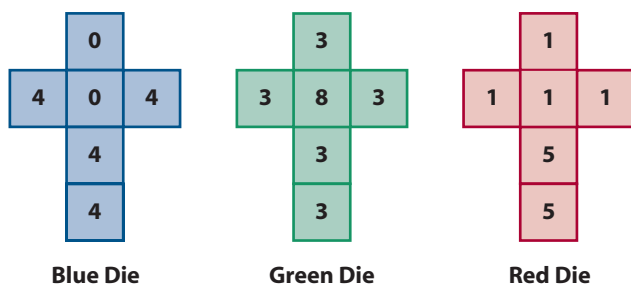


- 14 In the Think About This Situation at the beginning of this lesson, you were asked to consider the probability of rolling doubles on your first turn or on your second turn.
- a. Is it true that the probability of rolling doubles on your first turn or your second turn is $\frac{1}{6} + \frac{1}{6}$? Give an explanation for your answer.
 - b. Is it true that the probability of rolling doubles on at least one of your first six turns is $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$?

- 15 Refer to Applications Task 6. The age group of 21–24 had only 6,382 drivers involved in fatal accidents. This is less than all age groups until age 55 and up. Does this mean the drivers in this age group are less likely to get in a fatal crash? Explain your answer.
- 16 Why does it make sense to title Investigation 2 “The Addition Rule” rather than “Addition Rules”?

Extensions

- 17 Suppose that you flip a coin four times and record head (H) or tail (T) in the order it occurs.
 - a. Make a list of all 16 possible outcomes.
 - b. Are these outcomes equally likely?
 - c. Make a table of the probability distribution of the number of heads.
 - d. What is the probability that you will get exactly 2 heads? At most 2 heads?
- 18 Flavia selects one of the special dice with faces shown below, and then Jack selects one of the remaining two.

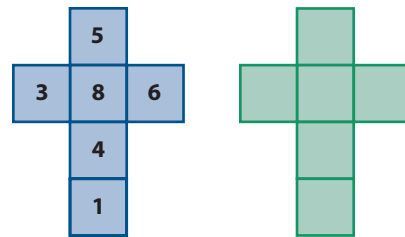


They each roll their die. The person with the larger number wins. To help you decide if it is better to use, for example, the blue die or the green die, you could complete a table like the following.

		Number on Green Die					
		3	3	3	3	3	8
Number on Blue Die	0	Green Die Wins					
	0	Green Die Wins					
	4	Blue Die Wins					
	4						
	4						
	4						

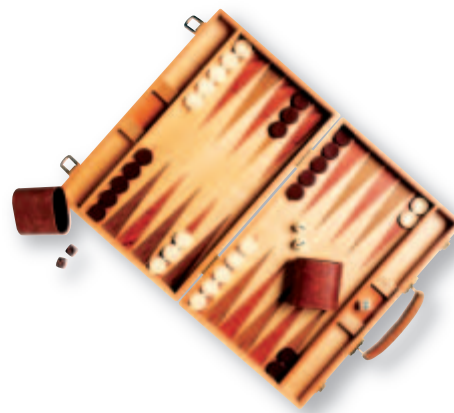
Suppose Flavia picks the blue die. To have the best chance of winning, which die should Jack choose? If Flavia picks the green die, which die should Jack choose? If Flavia picks the red die, which die should Jack choose? What is the surprise here?

- 19** Refer to the probability distribution table for the sum of two standard six-sided dice that you constructed in Problem 4 (page 534) of Investigation 1. Two nonstandard six-sided dice have this same probability distribution table. A net for one of those nonstandard dice is shown above. Using positive whole numbers, label a copy of the net for the other nonstandard die. The other die may be different from the one given, and numbers may be repeated on its faces.



- 20** Refer to the Venn diagrams you made in Connections Task 11. Use the appropriate one to help you write an Addition Rule that you can use to determine $P(A \text{ or } B \text{ or } C)$ when
- A and B are mutually exclusive, A and C are mutually exclusive, and B and C are mutually exclusive.
 - A and B are mutually exclusive, A and C are mutually exclusive, but B and C are not mutually exclusive.
 - A and B are not mutually exclusive, A and C are not mutually exclusive, and B and C are not mutually exclusive.

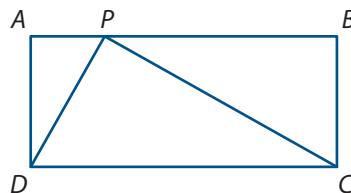
- 21** In the game of backgammon, if you “hit” your opponent’s checker exactly, that checker must go back to the beginning and start again. For example, to hit a checker that is three spaces ahead of your checker, you need to move your checker three spaces. You can do this either by getting a 3 on one die or by getting a sum of 3 on the two dice. In addition, if you roll double 1s, you can also hit your opponent’s checker because if you roll doubles, you get to move the numbers that show on the die twice each. So you move one space, then one space again, then one more space, hitting your opponent’s checker, then move your last space. (If you roll, say, double 4s, you can’t hit your opponent’s checker—you must skip over it as you move your first four spaces.)



- What is the probability of being able to hit a checker that is three spaces ahead of you?
- What is the probability of being able to hit a checker that is five spaces ahead of you?
- What is the probability of being able to hit a checker that is twelve spaces ahead of you?
- If you want to have the best chance of hitting a checker ahead of you, how many spaces should it be in front of you?

Review

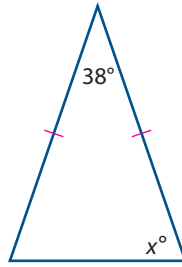
- 22** Determine if it is possible to draw zero, one, or more than one triangle that will fit the given description. Explain your reasoning.
- triangle XYZ with $m\angle X = 120^\circ$, $m\angle Y = 30^\circ$, and $XY = 8$ cm
 - a triangle with two right angles
 - an isosceles triangle with legs of length 5 cm and base of length 4 cm
 - a triangle with sides of length 2 in., 3 in., and 5 in.
 - a triangle with angles measuring 45° , 65° , and 70°
 - a right triangle with sides of length 10 m, 6 m, and 7 m
- 23** The mean number of people who attended five high school basketball games was 468. The tickets cost \$3 per person. What is the total amount received from ticket sales?
- 24** The height in feet of a punted football t seconds after a punt can be found using the equation $h = 1.8 + 50t - 16t^2$.
- How long was the football in the air?
 - How high did the football go?
 - When was the football 20 feet above the ground?
- 25** Solve each inequality and graph the solution on a number line. Substitute at least one number from your solution set back into the inequality to check your work.
- $3x + 7 \leq -59$
 - $16 \geq 12 - 2x$
 - $25 < 5(2x + 12)$
 - $4x + 18 > 6x - 24$
 - $\frac{1}{6}(5x + 12) - 8 \geq 0$
- 26** In the rectangle shown, is the sum of the areas of $\triangle APD$ and $\triangle BCP$ greater than, less than, or equal to the area of $\triangle DPC$? Justify your answer.



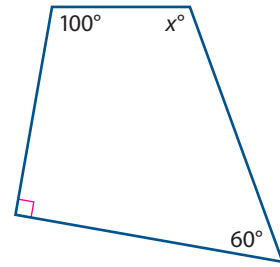
- 27** Find the following sums without using your calculator.
- $0.4 + 0.23$
 - $0.05 + 0.24 + 0.15$
 - $-0.62 + 0.82$
 - $\frac{1}{6} + \frac{2}{3}$
 - $\frac{2}{10} + \frac{1}{4} - \frac{3}{5}$

28 Find the value of x in each polygon below.

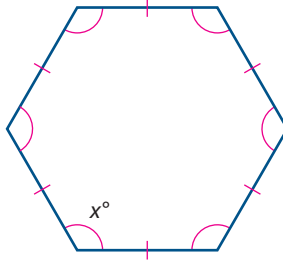
a.



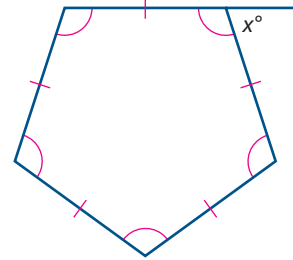
b.



c.



d.



29 Find two fractions on a number line that are between the two given fractions.

a. $\frac{1}{3}$ and $\frac{1}{2}$

b. $\frac{5}{6}$ and $\frac{6}{7}$

c. $\frac{a}{b}$ and $\frac{c}{d}$

LESSON 2



Modeling Chance Situations

In some cultures, it is customary for a bride to live with her husband's family. As a result, couples who have no sons and whose daughters all marry will have no one to care for them in their old age.

In 2000, China had a population of over 1,200,000,000. In an effort to reduce the growth of its population, the government of China had instituted a policy to limit families to one child. The policy has been very unpopular among rural Chinese who depend on sons to carry on the family farming and care for them in their old age.

Think About This Situation

Customs of a culture and the size of its population often lead to issues that are hard to resolve. But probability can help you understand the consequences of various policies.

- a** In a country where parents are allowed to have only one child, what is the probability that their one child will be a son? What is the probability they will not have a son? What assumption(s) are you making when you answer these questions?
- b** If each pair of Chinese parents really had only one child, do you think the population would increase, decrease, or stay the same? Explain your reasoning.
- c** Describe several alternative plans that the government of China might use to control population growth. For each plan, discuss how you might estimate the answers to the following questions.
 - i.** What is the probability that parents will have a son?
 - ii.** Will the total population of China grow, shrink, or stay about the same?
 - iii.** Will China end up with more boys than girls or with more girls than boys?
 - iv.** What is the mean number of children per family?

In this lesson, you will learn to estimate probabilities by designing simulations that use random devices such as coin flipping or random digits. In our complex world, simulation is often the only feasible way to deal with a problem involving chance. Simulation is an indispensable tool to scientists, business analysts, engineers, and mathematicians. In this lesson, you will also explore how geometric models can be used to solve probability problems.

Investigation 1 When It's a 50-50 Chance

Finding the answers to the questions in Part c of the Think About This Situation may be difficult for some of the plans you proposed. If that is the case, you can estimate the effects of the policies by creating a mathematical model that **simulates** the situation by copying its essential characteristics. Although slightly more than half of all births are boys, the percentage is close enough to 50% for you to use a probability of $\frac{1}{2}$ to investigate different plans. As you work on the problems of this investigation, look for answers to the following question:

How can you simulate chance situations that involve two equally likely outcomes?

- 1 Consider, first, the plan that assumes each family in China has exactly two children.
- Construct a sample space of the four possible families of two children.
 - Use your sample space to answer these questions from the Think About This Situation:
 - What is the probability that parents will have a son?
 - Will the total population of China grow, shrink, or stay about the same?
 - Will China end up with more boys than girls or with more girls than boys?
 - What is the mean number of children per family?



- 2 Your class may have discussed the following plan for reducing population growth in rural China.

*Allow parents to continue to have children until a boy is born.
Then no more children are allowed.*

Suppose that all parents continue having children until they get a boy. After the first boy, they have no other children. Write your best prediction of the answer to each of the following questions.

- In the long run, will the population have more boys or more girls, or will the numbers be approximately the same?
- What will be the mean number of children per family?
- If all people pair up and have children, will the population increase, decrease, or stay the same?
- What percentage of families will have only one child?
- What percentage of the children will belong to single-child families?



- 3 To get a good estimate of the answers to the questions in Problem 2, you could simulate the situation. To do this, you can design a **simulation model** that imitates the process of parents having children until they get a boy.
- Describe how to use a coin to conduct a simulation that models a family having children until they get a boy.
 - When you flip a coin to simulate one family having children until they get the first boy, you have conducted one **run** of your simulation. What is the least number of times you could have to flip the coin on a run? The most? If it takes n flips to get the first “boy”, how many “girls” will be in the family?

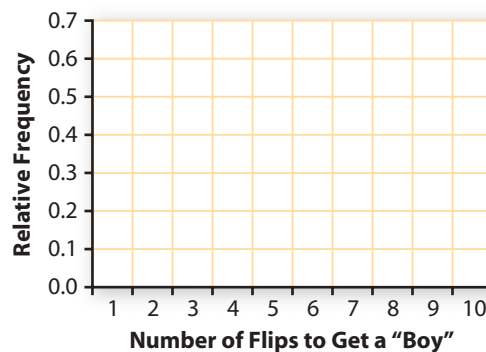
- c. Carry out one run of your simulation of having children until a boy is born. Make a table like the one below. Make a tally mark (/) in the tally column opposite the number of flips it took to get a “boy.”



Number of Flips to Get a “Boy”	Tally	Frequency (Number of Tallies)	Relative Frequency
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
⋮			
Total	200 runs	200 runs	1.00

- d. Continue the simulation until your class has a total of 200 “families.” Record your results in the frequency table. Add as many additional rows to the table as you need.
- e. How many “boys” were born in your 200 “families?” How many “girls?”
- f. What assumption(s) are you making in this simulation?
- g. Make a relative frequency histogram on a copy of the graph shown below and describe its shape.

Simulation of Number of Children

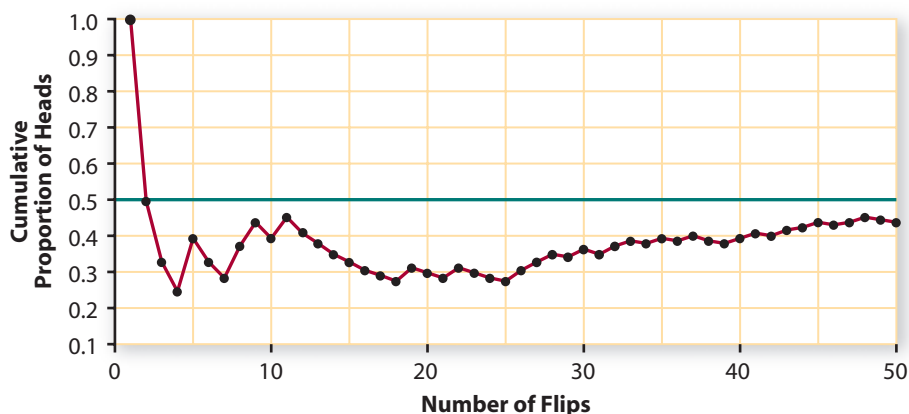


- h. Compare the median number of children to the mean number of children in the families.

- 4 Now reconsider the questions from Problem 2 which are reproduced below. Estimate the answers to these questions using your completed table from Problem 3.
- In the long run, will the population have more boys or more girls, or will the numbers be approximately the same?
 - What will be the mean number of children per family?
 - If all people pair up and have children, will the population increase, decrease, or stay the same?
 - What percentage of families will have only one child?
 - What percentage of the children will belong to single-child families?
- 5 Compare your estimates in Problem 4 with your original predictions in Problem 2. For which questions did your initial prediction vary the most from the estimate for the simulation? (If most of your original predictions were not accurate, you have a lot of company. Most people aren't very good at predicting the answers to probability problems. That's why simulation is such a useful tool.)

In the previous simulation, your class was asked to produce 200 runs. There is nothing special about that number except that it is about the most that it is reasonable for a class to do by hand. The **Law of Large Numbers** says that the more runs there are, the better your estimate of the probability tends to be. For example, according to the Law of Large Numbers, when you flip a coin a large number of times, eventually you should get a proportion of heads that is close to 0.5. (This assumes that the flips are independent and the probability of a head on each flip is 0.5.)

- 6 To illustrate the Law of Large Numbers, some students made the graph below. They flipped a coin 50 times and after each flip recorded the *cumulative* proportion of flips that were heads.



Source: Tim Erickson, *Fifty Fathoms*, Key Curriculum, 2002, pages 48–49.

- Was the first flip heads or tails? How can you tell from the graph whether the next flip was a head or a tail? What were the results of each of the first 10 coin flips?
- To three decimal places, what was the proportion of heads at each step in the first 10 coin flips?
- Could the graph eventually go back above 0.5? Explain.

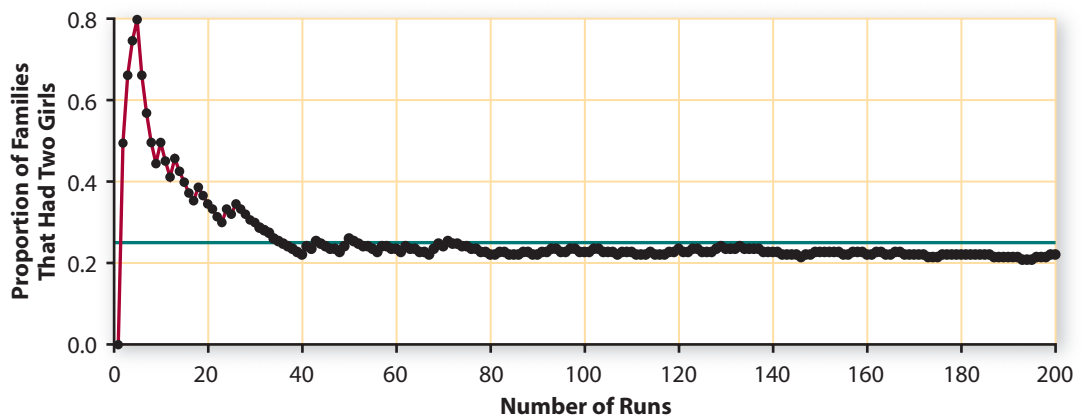
- d. Why do the lengths of the line segments between successive flips tend to get smaller as the number of flips increases?
- e. At the end of 50 flips, there were 22 heads and 28 tails. Continue on from there, doing 20 more flips. As you go along, complete a copy of the following table and a copy of the graph on the previous page, extending them to 70 flips.

Number of Flips	Frequency of Heads	Proportion of Heads
10	4	0.400
20	6	0.300
30	11	0.367
40	16	0.400
50	22	0.440
60		
70		

- f. Explain how your completed graph and table illustrate the Law of Large Numbers.

More flips of the coin or more runs of a simulation are better when estimating probabilities, but in practice it is helpful to know when to stop. You can stop when the distribution stabilizes; that is, you can stop when it seems like adding more runs isn't changing the proportions very much.

- 7 Refer to your graph showing the results of your coin flips from Problem 6. Does the proportion of heads appear to be stabilizing, or is there still a lot of fluctuation at the end of 70 flips?
- 8 The graph below shows the results of 200 runs of a simulation of the plan for reducing population growth in Problem 1. In that plan, each family has exactly two children. The proportion of families that had two girls is plotted. How does this graph illustrate the Law of Large Numbers?



Summarize the Mathematics

In this investigation, you designed and analyzed a simulation using coin flips.

- a Describe what a simulation is.
- b Why is it important to conduct a large number of runs in a simulation?
- c Suppose one run of a simulation is to flip a coin until you get a head. After many runs, you count the total number of heads and the total number of tails in all runs. Will you tend to have a larger proportion of heads or a larger proportion of tails?

Be prepared to share your ideas and reasoning with the class.

✓ Check Your Understanding

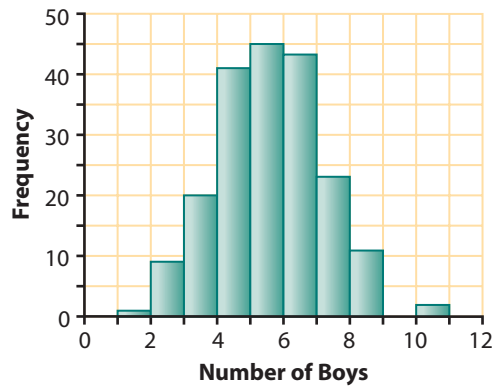
When asked in what way chance affected her life, a ninth-grader in a very large Los Angeles high school reported that students are chosen randomly to be checked for weapons. Suppose that when this policy was announced, a reporter for the school newspaper suspected that the students would not be chosen randomly, but that boys would be more likely to be chosen than girls. The reporter then observed the first search and found that all 10 students searched were boys.

- a. If there are the same number of boys and girls in the high school and a student is in fact chosen randomly, what is the probability that the student will be a boy?
- b. Write instructions for conducting one run of a simulation that models selecting 10 students at random and observing whether each is a boy or a girl.
- c. What assumptions did you make in your model?
- d. Perform five runs of your simulation and add your results to the frequency table at the right.

Number of Boys	Frequency (Before)	Frequency (After)
0	0	
1	1	
2	9	
3	20	
4	41	
5	45	
6	43	
7	23	
8	11	
9	0	
10	2	
Total	195 runs	200 runs



- e. The histogram below displays the results in the frequency table for 195 runs.



- i. Describe its shape and estimate its mean and standard deviation.
 - ii. Why do you think the bar should be a bit higher for 10 boys than for 9 boys?
 - iii. If you added your results to the histogram, would the basic shape change?
- f. Using results from 200 runs of your simulation, estimate the probability that all 10 students will be boys if students are selected at random to be searched. What conclusion should the reporter make?
- g. How much did adding your five runs change the probability of getting 10 boys? What can you conclude from this?

Investigation 2 Simulation Using Random Digits

In the previous investigation, the problem situations were based on two equally likely outcomes that you could simulate with a coin flip. In other situations, it is possible that there are more than two outcomes or the two outcomes aren't equally likely. In these situations, you can use random digits in designing a simulation. You can get strings of random digits from your calculator or from a random digit table produced by a computer. As you work on the following problems, look for answers to this question:

How can you use random digits when designing simulations?

- 1 Examine this table of random digits generated by a computer.

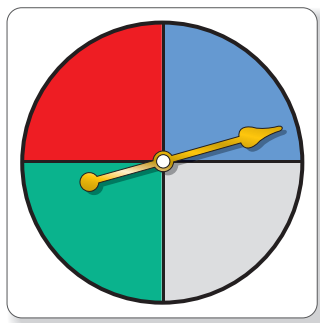
2 4 8 0 3	1 8 6 5 6	4 2 0 3 0	9 1 4 9 6
7 6 8 6 3	0 5 6 8 2	5 0 7 4 5	6 7 3 6 3
0 9 5 8 1	7 3 0 9 9	8 7 7 7 7	1 6 2 7 2
0 2 6 8 6	2 5 5 4 1	5 9 8 1 0	1 5 2 9 7
4 1 2 9 0	8 6 7 0 3	3 8 2 5 1	8 4 1 4 1
1 5 8 0 9	5 7 3 5 6	5 0 2 0 3	6 6 5 0 3
9 7 6 2 5	9 2 6 3 5	0 3 1 9 3	9 7 2 6 3
2 1 0 9 6	0 1 8 5 5	2 2 6 8 6	0 6 6 6 3

- a. How many digits are in the table? About how many 6s would you expect to find? How many are there?

- b. About what percentage of digits in a large table of random digits will be even?
- c. About what percentage of the 1s in a large table of random digits will be followed by a 2?
- d. About what percentage of the digits in a large table of random digits will be followed by that same digit?

- 2** Explain how you can use the table of random digits in Problem 1 to simulate each situation given below. (You may have to disregard certain digits.) Then perform one run of your simulation.
- a. Flip a coin and see if you get heads or tails.
 - b. Observe five coin flips and record how many heads you get.
 - c. Observe whether it rains or not on one day when the prediction is 80% chance of rain.
 - d. Select three cars at random from a large lot where 20% of the cars are black, 40% are white, 30% are green, and 10% are silver, and record the color.

- e. Spin the spinner shown here four times and record the colors.
- f. Roll a die until you get a 6 and record the number of rolls you needed.
- g. Select three different students at random from a group of ten students. How is this problem different from the others you have done?



- h. Select three different students at random from a group of seven students.

- 3** Suppose three meteorites are predicted to hit the United States. You are interested in how many will fall on publicly owned land. About 30% of the land in the United States is owned by the public.

- a. Describe how to conduct one run that simulates this situation. What assumptions are you making in this simulation?
- b. Combine results with your class until you have 100 runs and place the results in a frequency table that shows how many of the three meteorites fell on public land.
- c. What is your estimate of the probability that at least two of the meteorites will fall on public land?

- 4** In trips to a grocery store, you may have noticed that boxes of cereal often include a surprise gift such as one of a set of toy characters from a current movie or one of a collection of stickers. Cheerios, a popular breakfast cereal, once included one of seven magic tricks in each box.

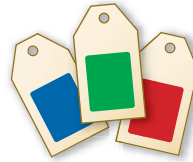
- a. What is the least number of boxes you could buy and get all seven magic tricks?

- b. If you buy one box of Cheerios, what is the probability that you will get a multiplying coin trick? To get your answer, what assumptions did you make about the tricks?

Collect All 7 and Put On Your Own Magic Show!



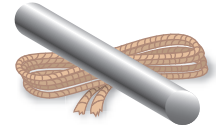
Money Clip Trick
Make two clips magically join together!



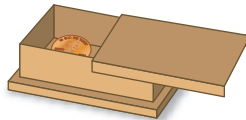
Mind-Reading Trick
Guess the color your friend secretly picked!



Vanishing Card Trick
Make a card magically disappear!



Magic Rope Trick
Make the rope magically pass through solid tube!



Disappearing Coin Trick
Make a coin magically disappear and reappear!



Surprise 4s
Turn two blank cards into two 4s!



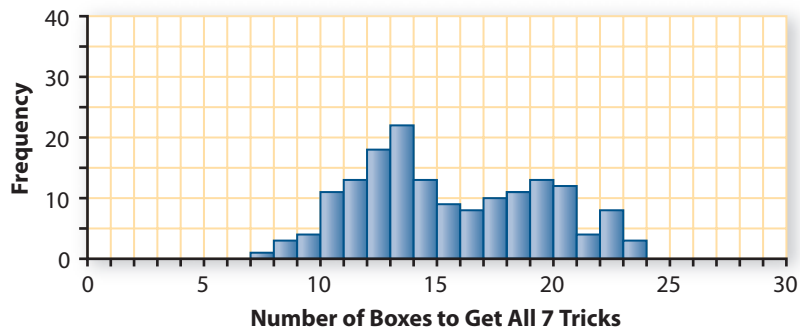
Multiplying Coin Trick
Turn two coins into three!

- c. Suppose you want to estimate the number of boxes of Cheerios you will have to buy before you get all seven magic tricks. Describe a simulation model, including how to conduct one run using a table of random digits. (Recall that you may ignore certain digits.)
- d. Compare your simulation model with that of other students. Then as a class, decide on a simulation model that all students will use.
- e. Perform five runs using the agreed upon simulation. Keep track of the number of “boxes” you would have to buy. Add your numbers and those for the other students in your class to a copy of this frequency table.

Number of Boxes to Get All 7 Tricks	Frequency (Before)	Frequency (After)
7	1	
8	3	
9	4	
10	11	
11	13	
12	18	
13	22	
14	13	
15	9	
16	8	

Number of Boxes to Get All 7 Tricks	Frequency (Before)	Frequency (After)
17	10	
18	11	
19	13	
20	12	
21	4	
22	8	
23	3	
24	0	
25	0	
⋮		
Total		

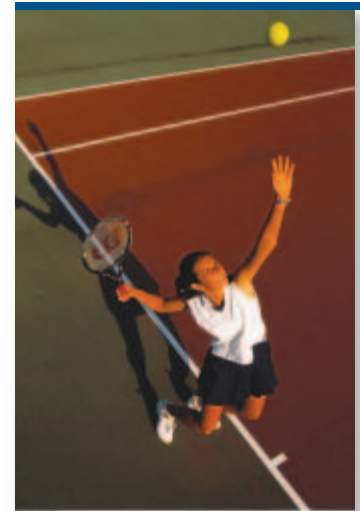
- f. Add the runs from your class to a copy of the histogram below.



- i. Describe the shape of the distribution.
- ii. Compare the shape of this distribution to others you have constructed in this unit.
- g. Based on the simulation, would it be unusual to have to buy 15 or more boxes to get the 7 magic tricks? Explain.
- h. What could be some possible explanations why a person would, in fact, end up buying a much larger number of boxes than you would expect from this simulation?

In the previous simulations, you have been able to use single digits from your table of random digits. In other cases, you may need to use groups of two or more consecutive random digits.

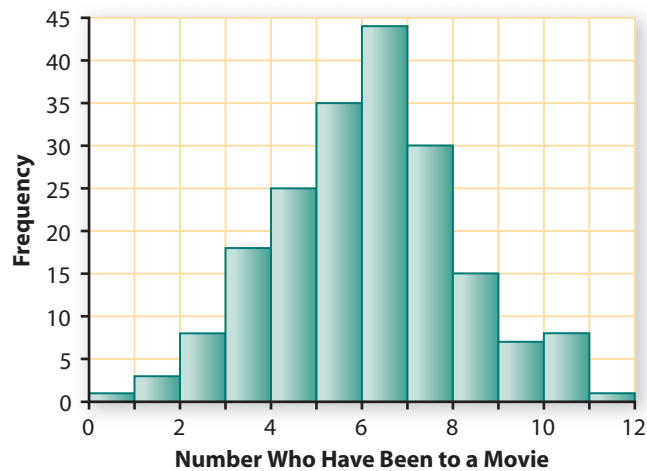
- 5 When playing tennis, Sheila makes 64% of her first serves. Describe how to use pairs of random digits to conduct one run of a simulation of a set where Sheila tries 35 first serves. Then conduct one run of your simulation and count the number of serves that Sheila makes.
- 6 Twenty-nine percent of the students at Ellett High School reported that they have been to a movie in the previous two weeks. Connie found that only 1 of her 20 (or 5%) closest friends at the school had been to a movie in the previous two weeks. Connie wants to know if this smaller-than-expected number can reasonably be attributed to chance or if she should look for another explanation.
- a. Describe how to use pairs of random digits to conduct one run of a simulation to find the number of recent moviegoers in a randomly selected group of 20 seniors.



- b. The table and histogram below show the results of 195 simulations of the number of recent moviegoers in groups of 20 randomly selected students. Conduct 5 runs, adding your results to a copy of the table and histogram.

Number of Students	Frequency (Before)	Frequency (After)
0	1	
1	3	
2	8	
3	18	
4	25	
5	35	

Number of Students	Frequency (Before)	Frequency (After)
6	44	
7	30	
8	15	
9	7	
10	8	
11	1	
Total	195	200



- c. Based on your simulation, estimate the probability that no more than 1 of 20 randomly selected students have been to a movie in the previous two weeks.
- d. Is the result for Connie's friends about what might be expected for a randomly selected group of 20 students? If not, what are some possible explanations?



A table of random digits is a convenient tool to use in conducting simulations. However, calculators and computer software with random number generators are more versatile tools.

- 7 Investigate the nature of the integers produced by the random integer generator on your calculator or computer software. You may find the command **randInt** under the probability menu. If, for example, you want six integers randomly selected from $\{1, 2, 3, 4, \dots, 19, 20\}$, enter **randInt(1,20,6)**.
- a. If the same integer can be selected twice, the random integer generator is said to select *with replacement*. If the same integer cannot appear more than once in the same set, the random integer generator is said to select *without replacement*. Does your **randInt** command select with replacement or without replacement?

b. Your calculator probably selects integers at random with replacement. If this is the case, describe how you can get a random selection of six numbers from $\{1, 2, 3, 4, \dots, 19, 20\}$ without replacement.

8 Explain how to use the **randInt** command to simulate each of the following situations. Then perform one run of the simulation.

a. Roll a die five times and record the number on top.

b. Flip a coin 10 times and record whether it is heads or tails on each flip.

c. Select three different students at random from a group of seven students.

d. Roll a die until you get a 6 and count the number of rolls needed.

e. Check five boxes of Cheerios for which of seven magic tricks they contain.

f. Draw a card from a well-shuffled deck of 52 playing cards and record whether it is an ace.

Summarize the Mathematics

In this investigation, you explored the properties of random digits and learned how to use them in designing a simulation.

- a What is a table of random digits? What command do you use to get random digits on your calculator?
- b Give an example of when you would want to select random digits with replacement. Without replacement.
- c How do you use random digits in a simulation when the probability of the event you want is 0.4? When the probability of the event is 0.394?
- d As a tie-in to the opening of a baseball season, each box of Honey Bunches of Oats cereal contained one of six Major League Baseball CD-ROMs (one for each division). Suppose you want to collect all six and wonder how many boxes you can expect to have to buy. How could you simulate this situation efficiently using a table of random digits? Using your calculator? What assumptions are you making?

Be prepared to share your ideas and reasoning with the class.



✓ Check Your Understanding

A teacher notices that of the last 20 single-day absences in his class, 10 were on Friday. He suspects that this did not happen just by random chance.

- a. Assuming that absences are equally likely to occur on each day of the school week, describe how to use a table of random digits to simulate the days of the week that 20 single-day absences occur.

- i. Conduct 5 runs. Add your results to a copy of the frequency table and histogram at the right that show the number of absences that are on Friday for 295 runs.

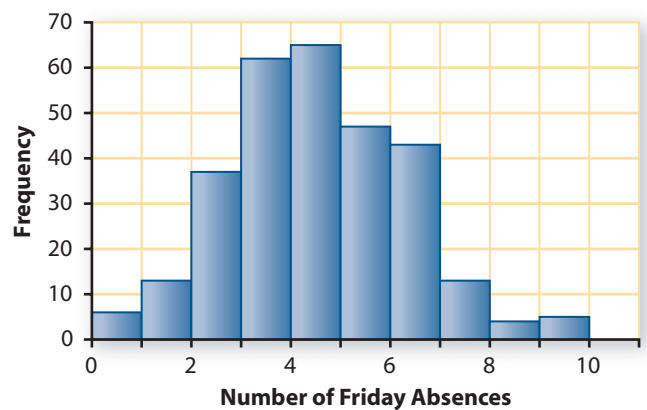
- ii. Based on the simulation, what is your estimate of the probability of getting 10 or more absences out of 20 on Friday just by chance? What should the teacher conclude?

- iii. From the simulation, what is your estimate of the mean number of absences on Friday, assuming that the 20 absences are equally likely to occur on each day of the week? Does this make sense? Why or why not?

- iv. How could you get better estimates for parts ii and iii?

- b. Describe how to use your calculator to conduct one run that simulates this situation.

Number of the 20 Absences that Are on Friday	Frequency (Before)	Frequency (After)
0	6	
1	13	
2	37	
3	62	
4	65	
5	47	
6	43	
7	13	
8	4	
9	5	
10	0	
Total	295	300



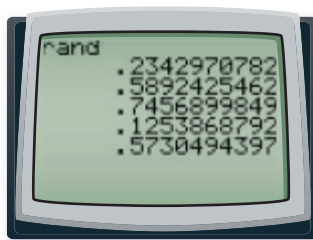
Investigation 3

Using a Random Number Generator

Sometimes a simulation requires that you select numbers from a continuous interval. For example, suppose you are painting a person on the backdrop in a school play. You can't decide how tall to make the person and so decide just to select a height at random from the interval 60 inches to 72 inches. There are infinitely many heights in that interval. For example, possible heights are 60 inches, 60.1 inches, 60.11 inches, 60.111 inches, 60.1111 inches, etc. As you work on problems in this investigation, look for answers to the following question:

How can you design simulations to solve problems when numbers can come from a continuous interval?

- 1 Investigate the nature of the numbers produced by selecting the command “rand” in the probability menu of your calculator and then pressing **ENTER** repeatedly.

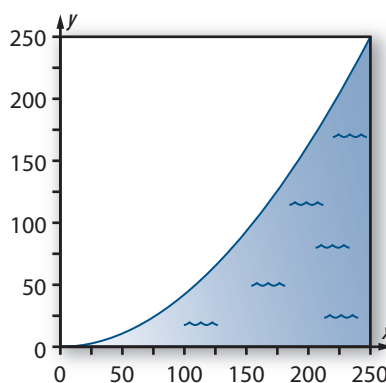


- a. How many decimal places do the numbers usually have? (Some calculators leave off the last digit if it is a 0.)
 - b. Between what two whole numbers do all the random numbers lie?
- 2 Now explore how to generate random numbers in other continuous intervals.
 - a. Generate random numbers of the form “6 rand” (or “6 × rand”). Between what two whole numbers do all the random numbers lie?
 - b. Between what two whole numbers do the random numbers lie when you use **10rand**? **36rand**? **rand+2**? **100rand+2**?
 - c. Write a **rand** command that selects a number at random from the interval between 1 and 7. Between 4 and 5. Between 60 inches and 72 inches.
 - d. Suppose you select two random numbers between 0 and 12 and want to estimate the probability that they are both more than 7.
 - i. Use your calculator to select two numbers from this interval. Record whether both numbers are more than 7.
 - ii. How can you simulate this situation using a spinner?
 - 3 Julie wakes up at a random time each morning between 6:00 and 7:00. If she wakes up after 6:35, she won't have time for breakfast before school. Conduct one run of a simulation to estimate the probability that Julie won't have time for breakfast the next two school days. Repeat this 10 times. What is your estimate of the probability that Julie won't have time for breakfast the next two school days?

- 4 Recall the triangle-building experiment in the *Patterns in Shape* unit. In this problem, suppose you have a 10-inch strand of uncooked spaghetti and select two places at random along its length. You break the strand at those places and try to make a triangle out of the three pieces.
- What do you think the probability is that the three pieces will form a triangle? Make a conjecture.
 - If the breaks are at 2-inch mark and 7.5-inch mark from the left end (see diagram below), how long are each of the three pieces? Can you make a triangle out of the three pieces?



- If the breaks are at 2.5-inch mark and at 6-inch mark from the left end, how long are each of the three pieces? Can you make a triangle out of the three pieces?
 - Use your calculator to simulate one run of the situation of breaking a 10-inch strand of spaghetti at two randomly selected places. Can you make a triangle out of the three pieces?
 - Carry out a simulation to estimate the probability that a triangle can be formed by the three pieces of a strand of spaghetti broken at two places at random along its length. Compare your estimated probability to your conjecture in Part a.
- 5 The square region pictured below is 250 feet by 250 feet. It consists of a field and a pond which lies below the graph of $y = 0.004x^2$. Imagine a totally inept skydiver parachuting to the ground and trying to avoid falling into the pond. The skydiver can be sure of landing somewhere inside the region, but the spot within it is random.



- Describe how to use the **rand** function of your calculator to simulate the point where the skydiver will land. (You will need an x -coordinate between 0 and 250 and a y -coordinate between 0 and 250.)
- How can you tell from the coordinates (x, y) of the simulated landing whether the skydiver landed in the pond?

- c. Simulate one landing and tell whether the skydiver landed in the pond or not.
- d. Simulate a second landing and tell whether the skydiver landed in the pond or not.

- 6 Al and Alice work at the counter of an ice cream store. Al takes a 10-minute break at a random time between 12:00 and 1:00. Alice does the same thing, independently of Al.
- a. Suppose that Al takes his break at 12:27. If Alice goes on her break at 12:35, would there be an overlap in the two breaks? What times for Alice to go on her break would result in an overlap of the two breaks?
 - b. Use the **rand** command to simulate this situation. Did the two breaks overlap?

Summarize the Mathematics

In this investigation, you explored further how to generate random numbers on your calculator and use them to estimate probabilities.

- a In what situations would you use a calculator command such as **rand** rather than a command such as **randInt**?
- b How would you use simulation to estimate the probability that two numbers randomly selected between 0 and 5 are both less than 3?

Be prepared to share your ideas and examples with the class.

✓ Check Your Understanding

Jerome arrives at school at a random time between 7:00 and 7:30. Nadie leaves independently of Jerome and arrives at a random time between 6:45 and 7:15. Suppose you want to estimate the probability that Nadie arrives at school before Jerome.

- a. Describe how to use a calculator command to simulate the time that Jerome arrives at school. To simulate the time that Nadie arrives at school.
- b. Perform one run of a simulation. Did Jerome or Nadie get to school first?
- c. Continue until you have a total of 10 runs. What is your estimate of the probability that Nadie gets to school before Jerome?

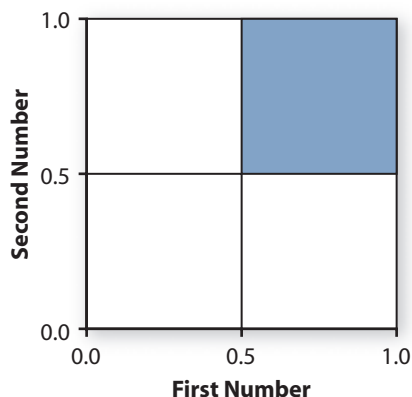


Investigation 4 Geometric Probability

You can solve some problems similar to those in the previous investigation without a simulation. The method involves drawing a geometric diagram and reasoning with areas. As you work on the problems in this investigation, look for answers to the following question:

How can you use area models to solve probability problems when numbers can come from a continuous interval?

- 1 Suppose you select two random numbers that are both between 0 and 1.
 - a. You want to find the probability that both numbers are more than 0.5. Explain how this problem can be solved using the diagram below.

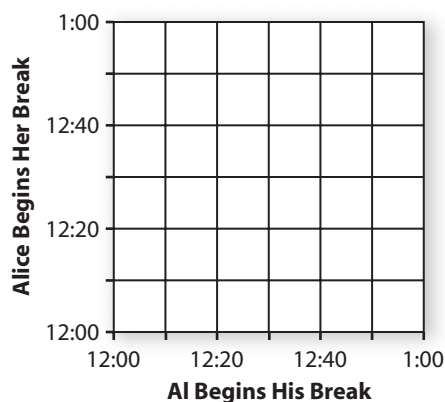


- b. Now suppose that you want to find the probability that both numbers are less than 0.2. Draw a square with sides of length 1 and shade in the area that represents the event that both numbers are less than 0.2. What is the probability that they are both less than 0.2?
 - c. Use an area model to find the probability that both numbers are less than 0.85.
- 2 Suppose you select two random numbers between 0 and 12 and want to find the probability that they both are greater than 7. Use an area model to represent this situation and find the probability.
 - 3 The National Sleep Association estimates that teens generally require 8.5 to 9.25 hours of sleep each night. However only 15% of teens report that they sleep 8.5 hours or more on school nights. (Source: www.sleepfoundation.org/_content/hottopics/sleep_and_teens_report1.pdf) Suppose you pick two teens at random and ask them if they sleep 8.5 hours or more on school nights.
 - a. Draw a square with sides of length 1 and shade in the area that represents the probability that at least one teen says that they sleep 8.5 hours or more on school nights.
 - b. Use your diagram from Part a to find the probability that at least one teen says they sleep 8.5 hours or more on school nights.



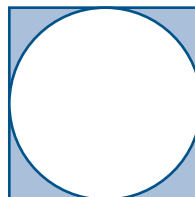
- 4** Suppose you select two random numbers between 0 and 1.
- Draw a square with sides of length 1 and shade in the area that represents the event that the sum of the two numbers is between 0 and 1.
 - What is the probability that their sum is less than 1?
 - What is the equation of the line that divides the shaded and unshaded regions in your area model?
 - Draw a square with sides of length 1 and shade in the area that represents the event that the sum of the two numbers is less than 0.6.
 - What is the probability that their sum is less than 0.6?
 - What is the equation of the line that divides the shaded and unshaded regions in this area model?

- 5** In Investigation 3 Problem 6 (page 566), you used simulation methods to estimate the probability that the breaks of two employees would overlap. The problem conditions were that Al takes a 10-minute break at a random time between 12:00 and 1:00 and Alice does the same thing, independently of Al. Now consider how you could calculate the exact probability using an area model.
- On a copy of the diagram below, identify points that correspond to identical start times for breaks by Al and Alice.

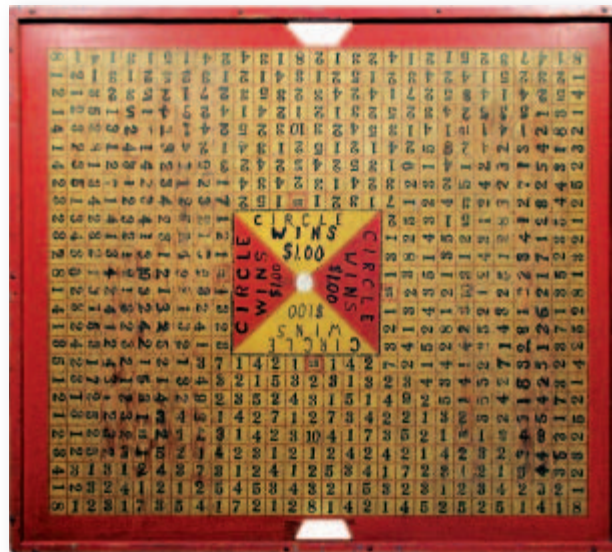


- For any given start time for a break by Al, what start times for a break by Alice would overlap with Al's break?
- On your diagram, shade in the beginning times of the two breaks that would result in overlap in their breaks.
- What is the probability that their two breaks overlap?

- 6** Imagine an archer shooting at the target shown. The square board has a side length of 6 feet. The archer can always hit the board, but the spot on the board is random. Use geometric reasoning to find the probability that an arrow lands in the circle.

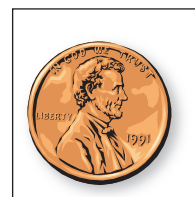


- 7 In an old carnival game, players toss a penny onto the surface of a table that is marked off in 1-inch squares. The table is far enough away that it is random where the penny lands with respect to the grid, but large enough that the penny always lands on the table.



If the penny lies entirely within a square, the player wins a prize. If the penny touches a line, the player loses his or her penny. A penny is about 0.75 inch in diameter.

- In order to win a prize, how far must the center of the penny be from any side of a 1-inch square?
- Draw a 1-inch square. Shade the region where the center of the penny must land in order to win a prize. What is the probability that the center of the penny lands in that region and the player wins the prize?



Summarize the Mathematics

In this investigation, you learned to use area models to find probabilities exactly.

- When can you use the geometric methods in this investigation to find probabilities?
- How would you use an area model to find the probability that two numbers randomly selected between 0 and 5 are both less than 3?

Be prepared to share your ideas and examples with the class.

✓ Check Your Understanding

The midpoint M of a 10-inch piece of spaghetti is marked as shown below. Suppose you break the spaghetti at a randomly selected point.



- What is the probability that the break point is closer to point M than to point A ?
- What is the probability that the break point is closer to point A or point B than to point M ?

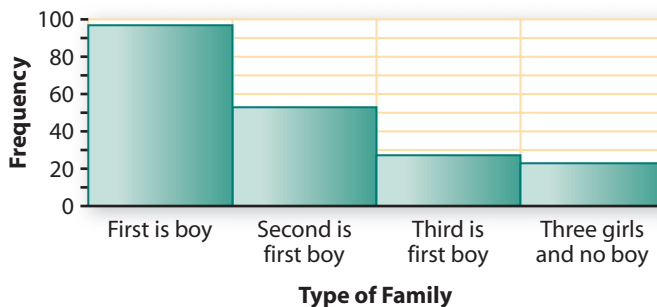
Applications

- 1 Suppose that a new plan to control population growth in China is proposed. Parents will be allowed to have at most three children and must stop having children as soon as they have a boy.
- Describe how to use a coin to conduct one run that models one family that follows this plan.
 - Conduct 5 runs. Copy the following frequency table, which gives the results of 195 runs. Add your results to the frequency table so that there is a total of 200 runs.

Type of Family	Frequency (Before)	Frequency (After)	Relative Frequency
First Child is a Boy	97		
Second Child is a Boy	50		
Third Child is a Boy	26		
Three Girls and No Boy	22		
Total	195	200	1.0



- Estimate the percentage of families that would not have a son.
- A histogram of the 195 results in the frequency table is given below. Describe its shape. If you added your results to the histogram, would the basic shape change? Explain your reasoning.



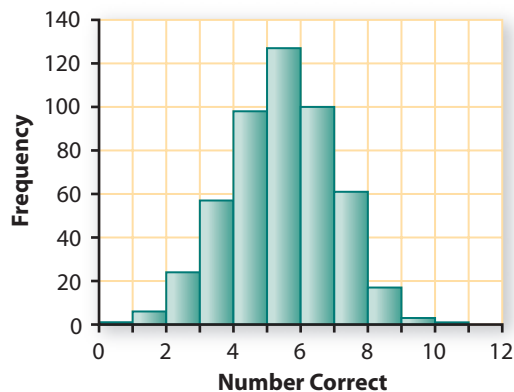
- How does the shape of the histogram based on 200 runs differ from that of the have-children-until-you-have-a-boy plan from Problem 3 of Investigation 1 on page 553? Explain why that makes sense.
- What is the mean number of children per family? Will the total population increase or decrease under this plan, or will it stay the same?
- In the long run, will this population have more boys or more girls, or will the numbers be about equal? Explain your reasoning.

2 Jeffrey is taking a 10-question true-false test. He didn't study and doesn't even have a reasonable guess on any of the questions. He answers "True" or "False" at random.

- a. Decide how to use a coin to conduct one run that models the results of this true-false test.
- b. Conduct 5 runs. Copy the following frequency table, which gives the results of 495 runs. Add your results to the frequency table so that there is a total of 500 runs.

Number Correct	Frequency (Before)	Frequency (After)
0	1	
1	6	
2	24	
3	57	
4	98	
5	127	
6	100	
7	61	
8	17	
9	3	
10	1	
Total Number of Runs	495	500

- c. A histogram of the 495 results in the frequency table is given below. Add your results to a copy of the histogram. Describe its shape and estimate its mean and standard deviation.



- d. On average, how many questions should Jeffrey expect to get correct using his random guessing method? How does this compare to the mean from the simulation?
- e. If 70% is required to pass the test, what is your estimate of the probability that Jeffrey will pass the test?
- f. Considering the Law of Large Numbers, should Jeffrey prefer a true-false test with many questions or with few questions?

3 The winner of baseball's World Series is the first of the two teams to win four games.

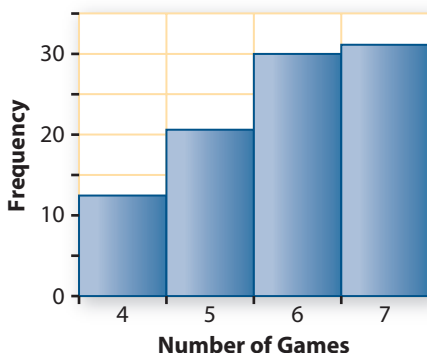
- a. What is the fewest number of games that can be played in the World Series? What is the greatest number of games that can be played in the World Series? Explain.
- b. Suppose that the two teams in the World Series are evenly matched. Describe how to use a table of random digits to conduct one run simulating the number of games needed in a World Series.



- c. Conduct 5 runs. Construct a frequency table similar to the one shown below and add your 5 results so that there is a total of 100 runs.

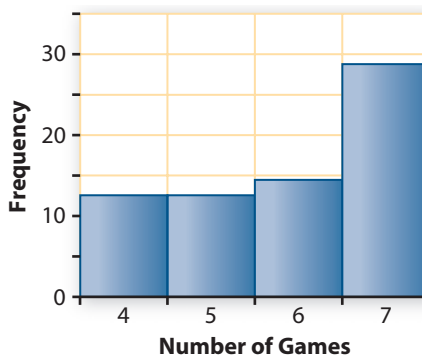
Number of Games Needed in the Series	Frequency (Before)	Frequency (After)
4	11	
5	21	
6	30	
7	33	
Total Number of Runs	95	100

- d. What is your estimate of the probability that the series will go seven games?
- e. By how much did your 5 results change the probability that the series will go seven games? What can you conclude from this?
- f. A histogram of the 95 results in the frequency table is given below. If you added your results to the histogram, how, if at all, would the basic shape change? Explain your reasoning.



- g. The following table and histogram give the actual number of games played in each World Series from 1940 through 2005. (There was no series in 1994.) Compare the results of the simulation with that of the actual World Series. What conclusions can you draw? (Source: www.infoplease.com/ippsa/A0112302.html)

Number of Games Played in the Series	Frequency
4	12
5	12
6	14
7	27



- 4 For some rock concerts, audience members are chosen at random to have their bags checked for cameras, food, and other restricted items. Suppose that you observe the first 25 males and 25 females entering a concert. Ten of them are chosen to be searched. All 10 are male. You will simulate the probability of getting 10 males if you randomly select 10 people from a group of 25 males and 25 females.

- a. When simulating the rock concert situation, why can't you use a flip of a coin like you did in the high school weapons search problem in the Check Your Understanding on page 557 of Investigation 1?

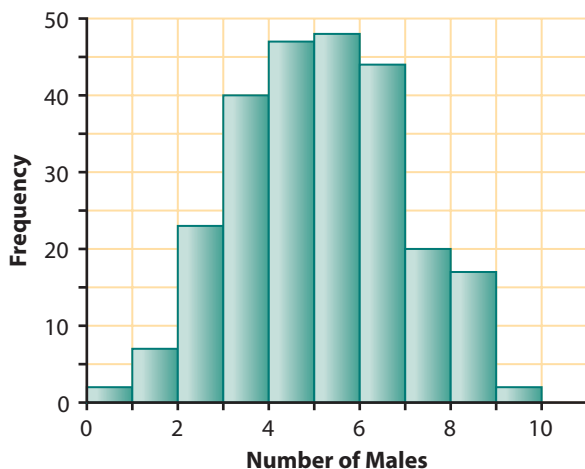
- b. Describe a simulation using slips of paper drawn from a bag. How are your assumptions different from those in the Check Your Understanding in Investigation 1 on page 557?

- c. How would you conduct this simulation using a table of random digits? Would you select with or without replacement?



- d. Conduct 5 runs of your simulation using the method of your choice. Record the results in a copy of the frequency table below, which shows the number of males selected.

Number of Males	Frequency (Before)	Frequency (After)
0	2	
1	7	
2	23	
3	40	
4	47	
5	48	
6	44	
7	20	
8	17	
9	2	
10	0	
Total Number of Runs	250	255



- e. What is your conclusion about the probability that all 10 concert attendees chosen to be searched would be male? Is your conclusion different from your conclusion in the high school weapons search problem? Explain.

5 According to the U.S. Department of Education report, *The Condition of Education 2003*, about 62% of high school graduates enroll in college immediately after high school graduation. (Source: nces.ed.gov/pubs2003/2003067_3.pdf)

- a. Describe how to use your random digit generator to simulate the situation of picking a randomly selected high school graduate and finding out if he or she enrolls in college immediately after graduation.

- b. Describe how to conduct one run of a simulation of the situation of selecting 30 high school graduates at random and counting the number who enroll in college immediately after graduation.
- c. Conduct 5 runs of your simulation. Copy the frequency table below that shows the results of 195 runs. Add your results to the table.

Number of Graduates Who Immediately Enroll in College	Frequency (Before)	Frequency (After)
10	1	
11	0	
12	1	
13	1	
14	6	
15	15	
16	13	
17	21	

Number of Graduates Who Immediately Enroll in College	Frequency (Before)	Frequency (After)
18	1	
19	0	
20	1	
21	1	
22	6	
23	15	
24	13	
25	21	
Total Number of Runs	195	200



- d. Use your results to estimate answers to these questions.
 - i. What is the probability that 12 or fewer of a randomly selected group of 30 high school graduates immediately enroll in college?
 - ii. Would it be unusual to find that 24 or more of 30 randomly selected graduates immediately enroll in college?
- e. Suppose you select a classroom of seniors at random in your school. You find that all 30 of the seniors in the class plan to enroll immediately in college. List as many reasons as you can why this could occur.

- 6 About 44.5% of violent crimes in the United States are committed by someone who is a stranger to the victim. (Source: *Statistical Abstract of the United States*, 2006, page 203, Table 311; www.census.gov/prod/www/statistical-abstract.html) Suppose that you select four violent crimes at random and count the number committed by strangers.
- a. Describe how to use the **randInt** function of your calculator to conduct one run that simulates this situation.
 - b. Describe how to use the **rand** function of your calculator to conduct one run that simulates this situation.
 - c. Conduct 10 runs using the calculator function of your choice and place the results in a frequency table that shows how many of the 10 violent crimes were committed by strangers.
 - d. What is your estimate of the probability that at least half of the four violent crimes were committed by strangers?

- 7 Suppose that you select two numbers at random from between 0 and 2. Draw a geometric diagram to help find the following probabilities.
- What is the probability that both are less than 1.2?
 - What is the probability that their sum is less than 1?
- 8 Jerome arrives at school at a random time between 7:00 and 7:30. Nadie leaves independently of Jerome and arrives at a random time between 6:45 and 7:15.
- Draw a geometric diagram to help find the probability that Nadie arrives at school before Jerome.
 - Compare your answers to that obtained in the Check Your Understanding on page 567.

Connections

- 9 You can use random devices other than coins to simulate situations with two equally likely outcomes.
- Is rolling the die below equivalent to flipping a coin? Explain.



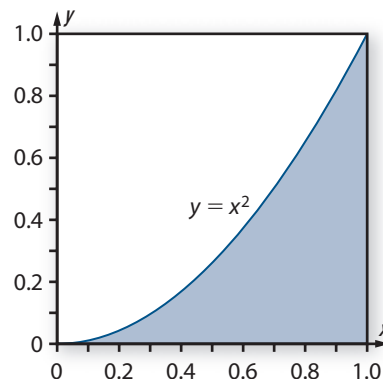
One View

Another View

- How could a regular, six-sided die marked with the numbers 1 through 6 be used to simulate whether a child is a boy or a girl?
 - How could a tetrahedral die marked with the numbers 1 through 4 be used to simulate whether a child is a boy or a girl?
 - Identify other geometric shapes of dice that could be used to simulate a birth.
- 10 Make an accurate drawing of a spinner that simulates rolling a die. Describe the characteristics of this spinner.
- 11 Refer to your frequency table from Problem 3 from Investigation 1 (page 553).
- Make a scatterplot of the ordered pairs (*number of flips to get a "boy," relative frequency*).
 - What function is a reasonable model of the relationship between *number of flips to get a "boy"* and *relative frequency*?
- 12 Explain how you can use each of the devices in the situation described.
- How could you use an icosahedral die to simulate picking random digits?

- b. How could you use a deck of playing cards to conduct one run in a simulation of collecting Cheerios tricks? (See Problem 4 on page 559 of Investigation 2.)
- c. How could you use a deck of playing cards to generate a table of random digits?

- 13 Describe how you could use simulation to estimate the area shaded in the diagram to the right. The area lies between the x -axis and the graph of $y = x^2$ on the interval $0 \leq x \leq 1$.



- 14 In Investigation 4, Problem 3 (page 568), you used geometric probability to find the probability that at least one of the two randomly selected teens says they sleep 8.5 hours or more on school nights. Dana suggested that since the probability for each teen reporting 8.5 or more hours of sleep is 15%, the probability that at least one teen says they sleep 8.5 hours or more is 30%, or 0.3. Use a geometric diagram and what you learned about the Addition Rule in Lesson 1 to explain why Dana’s thinking is incorrect.

Reflections

- 15 A school is selling magazine subscriptions to raise money. A group of students wants to simulate the situation of asking 10 people if they will buy a magazine and counting the number who will say yes. Jason proposes that the group flip a coin 10 times and count the number of heads since a person either says “yes” (heads) or says “no” (tails). Is Jason’s simulation model a reasonable one? Explain your position.
- 16 Suppose you want to estimate the probability that a family of two children will have at least one girl. You will count the number of heads in two flips of a coin. Does it matter if you flip *one* coin twice or flip *two* coins at the same time? Explain your reasoning.
- 17 The Law of Large Numbers tells you that if you flip a coin repeatedly, the percentage of heads tends to get closer to 50%. This is something that most people intuitively understand: the more “trials,” the closer $\frac{\text{number of heads}}{\text{number of flips}}$ should be to $\frac{1}{2}$.
- a. Jack flips a coin 300 times and gets 157 heads. What is his estimate of the probability that a coin will land heads?

- b. Julie flips a coin 30 times and uses her results to estimate the probability a coin will land heads. Would you expect Jack or Julie to have an estimate closer to the true probability of getting a head?
- c. Find the missing numbers in the table below. Do the results illustrate the Law of Large Numbers? Why or why not?

Number of Flips	Number of Heads	Percentage of Heads	Expected Number of Heads	Excess Heads
10	6	60	5	1
100	56	56	50	6
1,000	524			
10,000	5,140			

d. What surprising result do you see in the completed table?

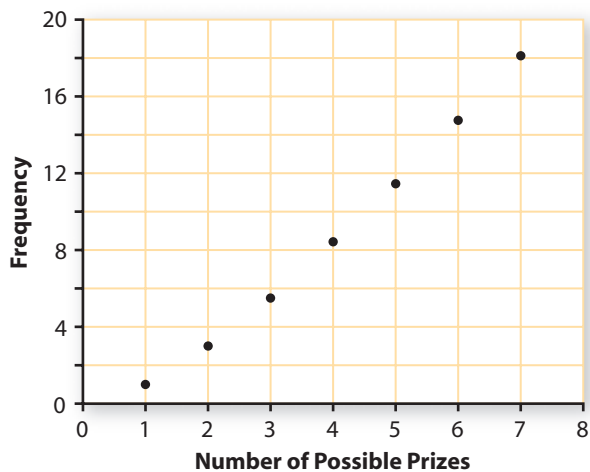
- 18 When presented with a problem involving chance, how would you decide whether to calculate the probability using a formula, using a geometric probability model, or by conducting a simulation?

Extensions

- 19 Suppose a cereal company is thinking about how many different prizes to use in its boxes. It wants children to keep buying boxes, but not get too discouraged. The company conducted very large simulations where all prizes are equally likely to be in each box. It could then estimate quite accurately the mean number of boxes that must be purchased to get all of the prizes. The company's estimates are given in the table below.

Number of Possible Prizes	Mean Number of Boxes Purchased
1	1
2	3
3	5.5
4	8.3
5	11.4
6	14.7
7	18.1

- a. Examine this scatterplot of (*number of prizes, mean number of boxes purchased*). Find a model that fits these data reasonably well.

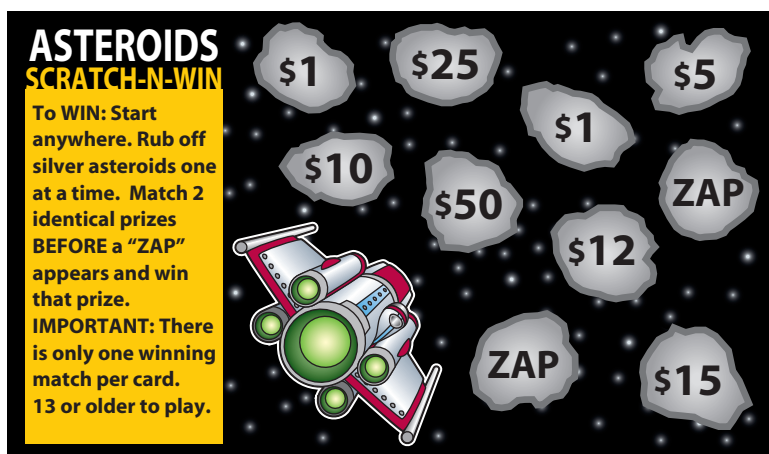


- b. Below is an exact formula that can be used to find the mean number of boxes that must be purchased when there are n equally likely prizes:

$$M(n) = n\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$$

- i. Check the values in the table using this formula.
 - ii. What is the mean number of boxes you would have to buy to get all of 20 possible prizes?
- 20 The card below shows a scratch-off game. There are ten asteroids. Two say “Zap,” two say “\$1,” and the other six name six larger cash prize amounts. All of the asteroids were originally covered. The instructions say:

Start anywhere. Rub off silver asteroids one at a time. Match 2 identical prizes BEFORE a “ZAP” appears and win that prize. There is only one winning match per card.



- a. Describe how to conduct one run of a simulation to estimate the probability of winning a prize with this card. Conduct 10 runs of your simulation and estimate the probability.
- b. Would the estimated probability be different if the prizes with no match weren't on the card?

- 21 Toni doesn't have a key ring and so just drops her keys into the bottom of her backpack. Her four keys—a house key, a car key, a locker key, and a key to her bicycle lock—are all about the same size.



- If she reaches into her backpack and grabs the first key she touches, what is the probability it is her car key?
- If the key drawn is not her car key, she holds onto it. Then, without looking, she reaches into her backpack for a second key. If that key is not her car key, she holds on to both keys drawn and reaches in for a third key. Do the chances that Toni will grab her car key increase, decrease, or remain the same as she continues? Explain your reasoning.
- The frequency table below gives the results of 1,000 simulations of this situation. From the frequency table, it appears that the numbers of keys needed are equally likely. Explain why this is the case.

Number of Grabs Toni Needs to Get Her Car Key	Frequency
1	255
2	252
3	236
4	257
Total Number of Runs	1,000

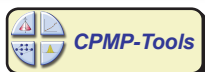
- Estimate the probability that all four keys have to be drawn before Toni gets her car key.

- 22 Toni's key selection problem in Extensions Task 21 is one that depends on *order*—the order in which she chooses the keys. One way to model the problem would be to list all the possible orders in which the keys could be selected. An ordering of a set of objects is called a **permutation** of the objects. For example, the six permutations of the letters A, B, and C are:

ABC ACB BAC BCA CAB CBA

- List all of the permutations of the letters A and B. How many are there?
- List all of the permutations of the letters A, B, C, and D. How many are there?
- Look for a pattern relating the number of permutations to the number of letters.

- d. How many permutations are there of Toni’s four keys? What is the probability that all four keys have to be drawn before Toni gets her car key? Compare your answer with that for Extensions Task 21 Part d.
- e. How many permutations do you think there are of the letters A, B, C, D, and E? Check your conjecture by using the permutations option on your calculator or computer software. (For most calculators, you need to enter “5 nPr 5”. This means the number of permutations of 5 objects taken 5 at a time.)



23 In the history of National Basketball Association finals, the home team has won about 71% of the games. Suppose that the Los Angeles Lakers are playing the Philadelphia 76ers in the NBA finals. The two teams are equally good, except for this home team advantage. The finals are a best-of-seven series. The first two games will be played in Philadelphia, the next three (if needed) in Los Angeles, and the final two (if needed) in Philadelphia.

- a. What is the probability that the 76ers will win a game if it is at home? What is the probability that the 76ers will win a game if it is played in Los Angeles? What is the probability the 76ers will win the first game of the series? The second game? The third game? The fourth game? The fifth game? The sixth game? The seventh game?
- b. Describe how to conduct one run to simulate this series.
- c. Conduct 5 runs and add your results to a copy of the frequency table below to make a total of 200 runs.

Number of Games Won by the 76ers	Frequency (Before)	Frequency (After)
0	9	
1	33	
2	21	
3	20	
4	112	
Total Number of Runs	195	200

- d. What is your estimate of the probability that the 76ers win the finals?
- e. Suppose that, to cut travel costs, the NBA schedules three games in Los Angeles followed by four in Philadelphia.
 - i. Design a simulation to estimate the probability that the 76ers win the finals in this situation.

- ii. Conduct 5 runs, adding your results to those in a copy of the frequency table below.

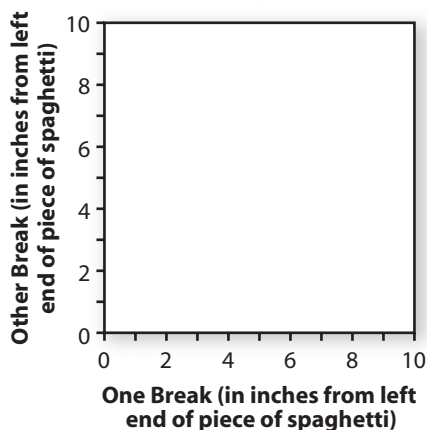
Number of Games Won by the 76ers	Frequency (Before)	Frequency (After)
0	21	
1	21	
2	22	
3	19	
4	112	
Total Number of Runs	195	200

- iii. What is your estimate of the probability that the 76ers win this series?

- f. Compare the estimated probability that the 76ers win the series in Parts d and e. Does it matter which way the series is scheduled?

24 Suppose that you break a 10-inch strand of spaghetti at two random places.

- a. On a copy of the diagram below, shade in the region that includes the points where a triangle can be formed from the three pieces.



- b. Find the probability that a triangle can be formed from the three pieces.

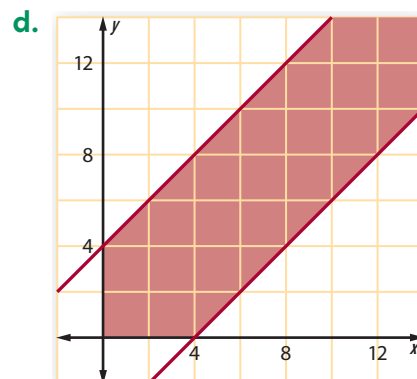
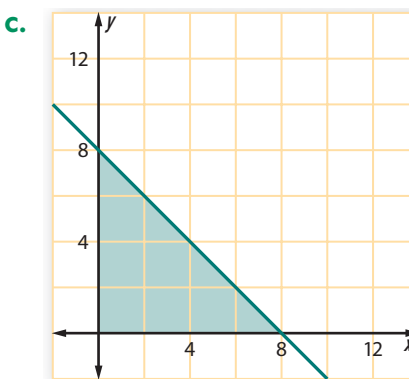
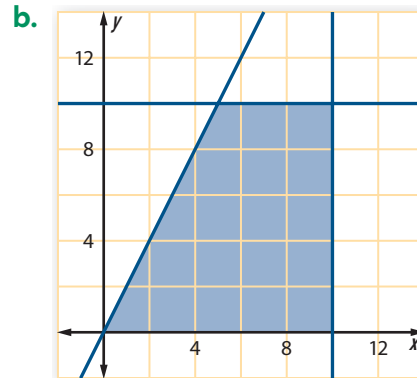
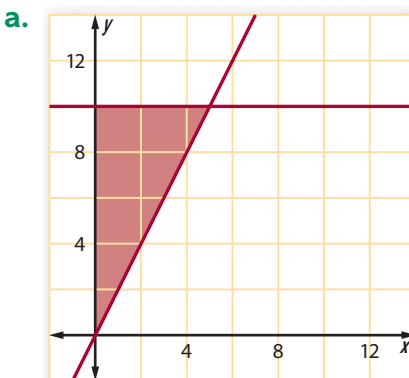
Review

25 Find the number that lies halfway between the two given numbers on a number line.

- a. 0.005 and 0.006
- b. 0.12345 and 0.123451
- c. 0.52 and 0.520001

- 26** In reporting health statistics like births, deaths, and illnesses, rates are often expressed in phrases like “26 per 1,000” or “266 per 100,000.” Express each of the following in equivalent percent language.
- Infant mortalities occur in about 7 of every 1,000 live births in the United States.
 - Each year about 266 out of every 100,000 Americans die from heart disease.
 - Twins, triplets, or other multiple births occur in about 255 out of every 10,000 births in the United States.

- 27** Find the area of the shaded portion on each graph below.



- 28** Graph each of the following lines on a separate coordinate system.
- $y = -2$
 - $y = 8 - \frac{5}{3}x$
 - $x + y = 4$
 - $x = 3$
 - $y = 2x - 3$

- 29** John could take any of four different routes to get from his home to school and then take any one of five different routes to get to after-school soccer practice. How many different ways are there for him to travel from home to school to soccer practice? Illustrate your answer using a geometric diagram.

- 30** For each of the tables of values below:
- Decide if the relationship between x and y can be represented by a linear, exponential, or quadratic function.
 - Find an appropriate function rule for the relationship.
 - Use your rule to find the y value that corresponds to an x value of 10.

a.

x	-2	-1	0	1	2	3	4	5
y	12	5	0	-3	-4	-3	0	5

b.

x	-2	-1	0	1	2	3	4	5
y	1	2.5	4	5.5	7	8.5	10	11.5

c.

x	-2	-1	0	1	2	3	4	5
y	1	2	4	8	16	32	64	128

- 31** Draw and label two right triangles PQR and WXY for which: $\angle Q$ and $\angle X$ are right angles, $\overline{PR} \cong \overline{WY}$, and $\angle R \cong \angle Y$. Explain as precisely as you can whether or not $\triangle PQR \cong \triangle WXY$.

- 32** Rewrite each expression in equivalent standard quadratic form, $ax^2 + bx + c$.

- | | |
|------------------------------|-----------------------------|
| a. $4x(6 - x)$ | b. $(x + 5)(x + 10)$ |
| c. $x - 2x(x + 3)$ | d. $(7x - 2)^2$ |
| e. $(3x + 6)(3x - 6)$ | f. $(3x + 5)(x - 9)$ |

- 33** Write each exponential expression in simplest possible equivalent form using only positive exponents.

- | | |
|--------------------------------|---------------------------------------------|
| a. $\frac{(2x)^3}{x^5}$ | b. $(-4a^3bc^4)^3$ |
| c. $(4x^3y)(-6x^4y)$ | d. $3x^{-2}$ |
| e. $(4n)^{-1}$ | f. $\left(\frac{6xy^2}{2y}\right)^3$ |

- 34** Rhombus $ABCD$ has sides of length 11 cm.
- Could the length of diagonal \overline{AC} be 6 cm? What about 24 cm? Explain your reasoning.
 - If $m\angle A = 50^\circ$, find the measures of the other three angles.
 - If rhombus $ABCD$ is a square, find the length of diagonal \overline{BD} .

Looking Back

When outcomes are equally likely, such as those from the roll of a die, you often can calculate probabilities exactly. In this unit, you learned how to write out sample spaces for such situations and to make probability distributions based on them. For more complex situations, you used geometric models to find probabilities or simulation to estimate probabilities. An important feature of your simulation models was the use of coins, dice, spinners, or random numbers to produce random outcomes. In each case, the simulation had the same mathematical characteristics as those of the original problem.

The tasks in this final lesson will help you review and apply key ideas for constructing probability distributions and using simulation and geometric models to solve problems involving chance.

- 1 Suppose that you roll two octahedral dice, which have the numbers 1 through 8 on each one.



- a. Make a sample space that shows all possible outcomes.
 - b. Make a probability distribution table for the sum of the two dice.
 - c. What is the probability that the sum is 8? At least 8?
 - d. Make a probability distribution table for the absolute value of the difference of the two dice.
 - e. What is the probability that the difference is 6? At most 6?
- 2 Show how to use the appropriate form of the Addition Rule to answer these questions about rolling two octahedral dice.
 - a. What is the probability you get doubles or a sum of 7?
 - b. What is the probability you get doubles or a sum of 8?
 - c. What is the probability you get a sum of 7 or a sum of 8?



3 According to the 2000 census, about 10% of the population of Los Angeles is African American.

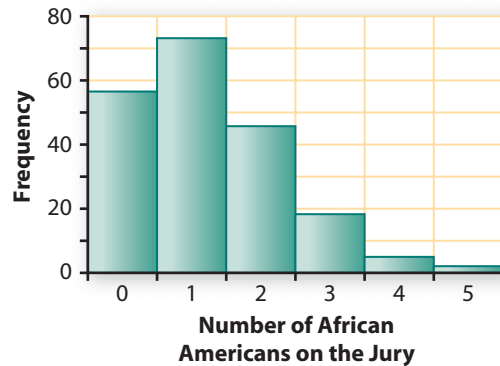
a. Juries have 12 members. Design a simulation model to determine the probability that a jury randomly selected from the population of Los Angeles would have no African American members.

b. Conduct 5 runs. Add your results to a copy of the frequency table below so that there is a total of 200 runs.



Number of African Americans on the Jury	Frequency
0	56
1	73
2	45
3	16
4	4
5	1
Total Number of Runs	200

c. Add your results to a copy of the histogram below. Describe its shape and estimate the mean.

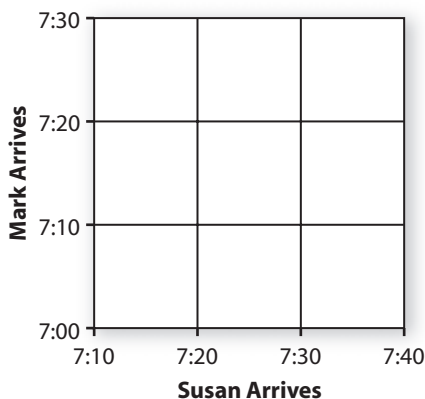


d. What is your estimate of the probability that a randomly selected jury of 12 people would have no African American members?

- e. A *grand jury* decides whether there is enough evidence against a person to bring him or her to trial. A grand jury generally consists of 24 people. Do you think the probability that a randomly selected grand jury in Los Angeles would have no African American members is more, less, or the same as your answer to Part d? Why?



- 4 This roller coaster has 7 cars. Ranjana stands in a long line to get on the ride. When she gets to the front, the attendant directs her to the next empty car. No one has any choice of cars, but must take the next empty one in the coaster. Ranjana goes through the line 10 times, each time hoping she gets to sit in the front car.
- Each time she goes through the line, what is the probability that Ranjana will sit in the front car? Do you think Ranjana has a good chance of sitting in the front car at least once in her 10 rides? Explain your reasoning.
 - Describe how to use random digits to conduct one run simulating this situation.
 - Perform 15 runs. Place the results in a frequency table that lists the number of times out of the 10 rides that Ranjana sits in the front car.
 - From your simulation, what is your estimate of the probability that Ranjana will sit in the front car at least once?
 - How could you get a better estimate of the probability that Ranjana will sit in the front car at least once?
- 5 Mark arrives at the gym at a random time between 7:00 and 7:30. Susan arrives at a random time between 7:10 and 7:40.
- Shade in the region on the following diagram that represents the event that Mark gets to the gym before Susan.



- What is the probability that Mark gets to the gym before Susan?
- What is the probability that Susan gets to the gym before Mark?

Summarize the Mathematics

In this unit, you learned how to find exact probabilities using a sample space of equally likely outcomes. You also learned how to use simulation and geometric diagrams to model more complex situations.

- a What is a probability distribution?
- b What are mutually exclusive events? Give an example of two mutually exclusive events. Give an example of two events that aren't mutually exclusive.
- c How can you find $P(A \text{ or } B)$ when A and B are mutually exclusive? When they aren't?
- d Summarize the steps involved in using simulation to model a problem involving chance.
- e Why doesn't simulation give you an "exact" probability? What does the Law of Large Numbers tell you about how to get a more precise answer?
- f Summarize the main ideas involved in using geometric probability models to solve problems involving chance.

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.