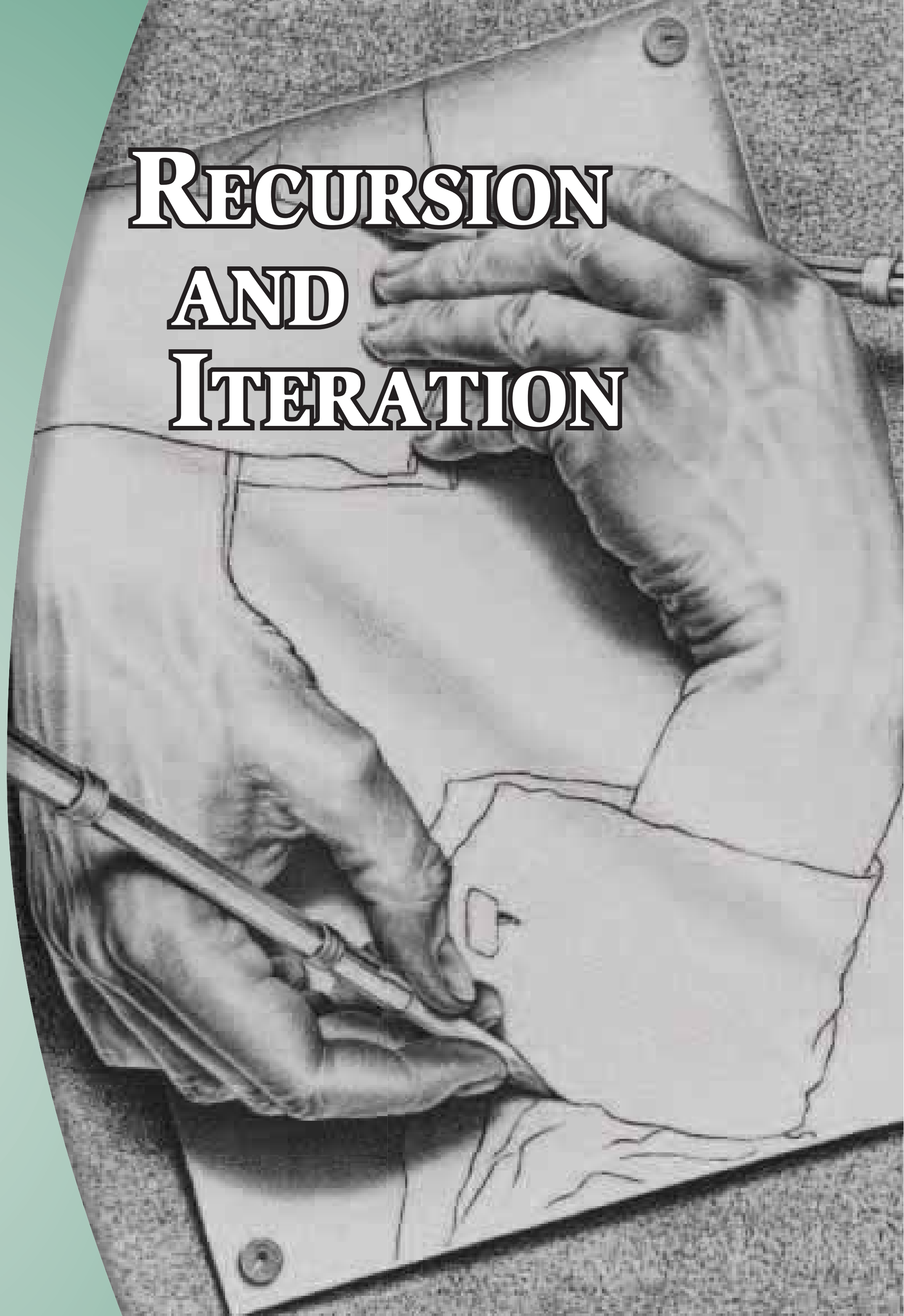
UNIT 7

Recursion and Iteration



In previous units, you have used equations, tables, and graphs to investigate linear, exponential, polynomial, and periodic patterns of change. You have used coordinates and matrices to model geometric change in position, size, and shape. In many situations, it is also important to understand step-by-step sequential change, such as yearly change in population or hourly change in antibiotic concentration after taking medication.

Recursion and iteration are powerful tools for studying sequential change. You have already used recursion and iteration when you used *NOW-NEXT* rules to solve problems. In this unit, you will study sequential change more fully. The concepts and skills needed are developed in the following three lessons.

Lessons

1 *Modeling Sequential Change Using Recursion and Iteration*

Represent and solve problems related to sequential change, using subscript and function notation and technological tools, such as spreadsheets.

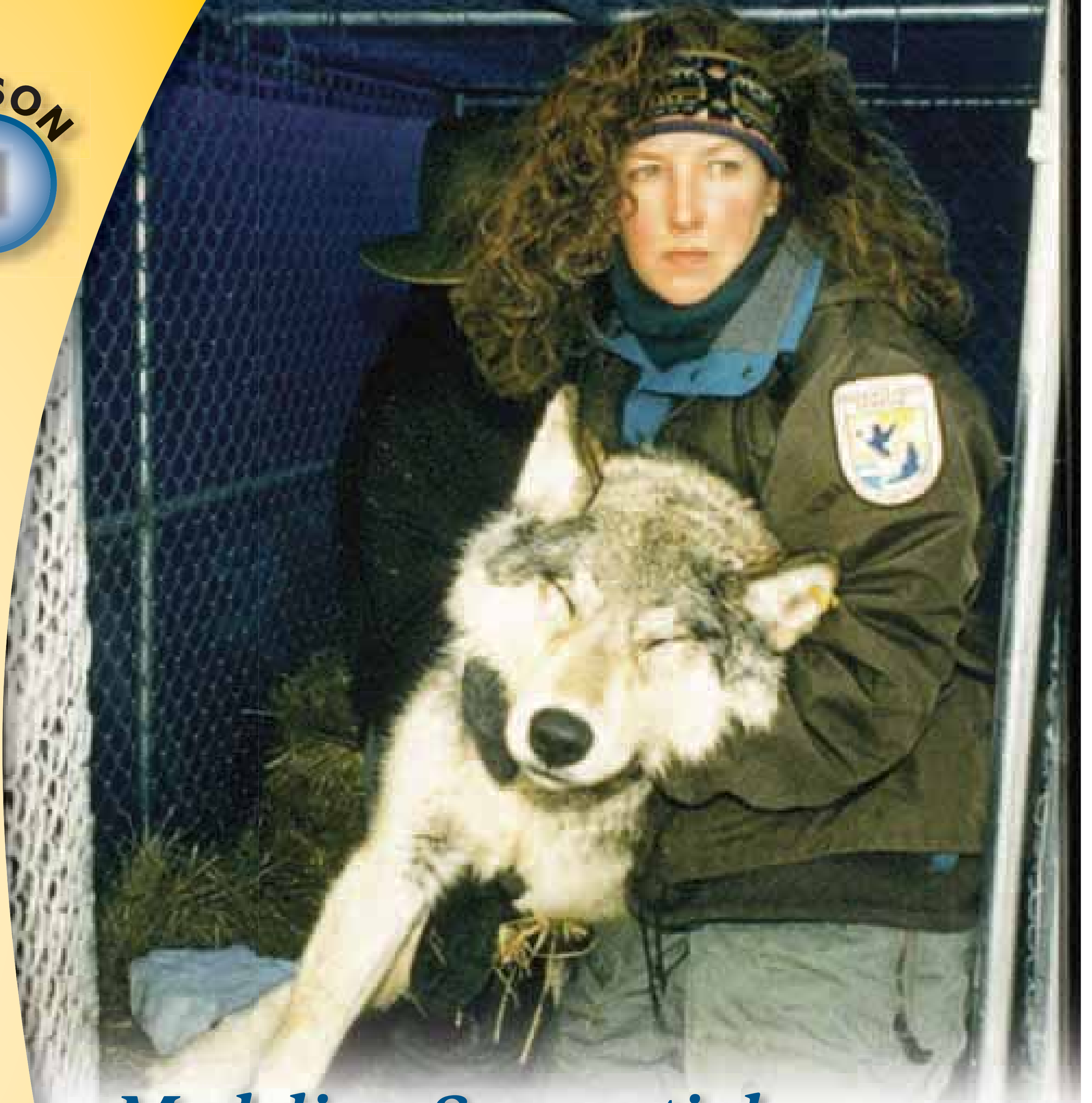
2 *A Recursive View of Functions*

Analyze linear, exponential, and polynomial functions from a recursive point of view, specifically through the study of arithmetic and geometric sequences and finite differences tables.

3 *Iterating Functions*

Investigate the general process of iterating functions, and completely analyze the behavior of linear functions when they are iterated, including an analysis of slope to identify attracting and repelling fixed points.

LESSON 1



Modeling Sequential Change Using Recursion and Iteration

Wildlife management has become an increasingly important issue as modern civilization puts greater demands on wildlife habitat. Tracking annual changes in the size of wildlife population is essential to effective management. In previous units, you have used *NOW-NEXT* rules or formulas to model situations involving sequential change. Now you will examine sequential change more fully.

Think About This Situation

Suppose you are in charge of managing the fish population in a pond that is stocked with fish from a Department of Natural Resources hatchery.

a. What are some factors to consider in managing the fish population?

b. How could you estimate the current fish population?

c. Why would it be useful to be able to predict the year-to-year changes in the fish population? Why would knowledge of the long-term population changes be useful?

In this unit, you will use recursion and iteration to represent and solve problems related to sequential change, such as year-to-year change in a population or month-to-month change in the amount of money owed on a loan. You will use subscript notation and function notation. Spreadsheet software will be used to help with the analysis.

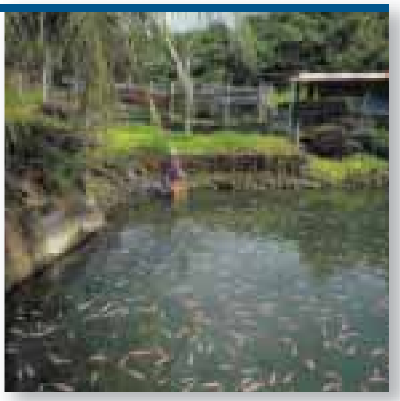
Investigation 1 Modeling Population Change

The first step in analyzing sequential change situations, like the fish population situation, is to build a mathematical model. As you work on the problems of this investigation, look for answers to this question:

How can you construct and use a mathematical model to help you analyze a changing fish population?

As you have seen before, a typical first step in mathematical modeling is simplifying the problem and deciding on some reasonable assumptions. Three factors that you may have listed in the Think About This Situation discussion are initial fish population in the pond, annual growth rate of the population, and annual restocking amount, that is, the number of fish added to the pond each year. For the rest of this investigation, use just the following assumptions.

- There are 3,000 fish currently in the pond.
- 1,000 fish are added at the end of each year.
- The population decreases by 20% each year (taking into account the combined effect of all causes, including births, natural deaths, and fish being caught).



1 Using these assumptions, build a mathematical model to analyze the population growth in the pond as follows.

a. Estimate of the population after one year. Estimate the population after two years. Describe how you computed these estimates. What additional details did you assume to get the answers that you have?

b. Assume that the population decreases by 20% *before* the 1,000 new fish are added. Also assume that the population after each year is the population *after* the 1,000 new fish are added. Are these the assumptions you used to compute your answers in Part a? If not, go back and recompute using these assumptions. Explain why these assumptions are reasonable. These are the assumptions you will use for the rest of this analysis.

c. Write a formula using the words *NOW* and *NEXT* to model this situation as specified in Part b.

d. Use the formula from Part c and the last-answer feature of your calculator or computer software to find the population after seven years. Explain how the keystrokes or software features you used correspond to the words *NOW* and *NEXT* in the formula.

2 Now think about the patterns of change in the long-term population of fish in the pond.

a. Do you think the population will grow without bound? Level off? Die out? Make a quick guess about the long-term population. Compare your guess to those made by other students.

b. Determine the long-term population by continuing the work that you started in Part d of Problem 1.

c. Explain why the long-term population you have determined is reasonable. Give a general explanation in terms of the fishing pond ecology. Also, based on the assumptions above, explain mathematically why the long-term population is reasonable.

d. Does the fish population change faster around year 5 or around year 25? How can you tell?

3 What do you think will happen to the long-term population of fish if the initial population is different but all other conditions remain the same? Make an educated guess. Then check your guess by finding the long-term population for a variety of initial populations. Describe the pattern of change in long-term population as the initial population varies.

4 Investigate what happens to the long-term population if the annual restocking amount changes but all other conditions are the same as in the original assumptions. Describe as completely as you can the relationship between long-term population and restocking amount.

5 Describe what happens to the long-term population if the annual decrease rate changes but all other conditions are the same as in the original assumptions. Describe the relationship between long-term population and the annual decrease rate.

6 Now consider a situation in which the fish population shows an annual rate of *increase*.

a. What do you think will happen to the long-term population if the population *increases* at a constant annual rate? Make a conjecture and then test it by trying at least two different annual increase rates.

b. Write formulas using *NOW* and *NEXT* that represent your two test cases.

c. Do you think it is reasonable to model the population of fish in a pond with an annual rate of increase? Why or why not?

Summarize the Mathematics

In this investigation, you constructed a *NOW-NEXT* formula to model and help analyze a changing fish population. Consider a population change situation modeled by *NEXT* = 0.6*NOW* + 1,500.

a Describe a situation involving a population of fish that could be modeled by this *NOW-NEXT* formula.

b What additional information is needed to be able to use this *NOW-NEXT* formula to predict the population in 3 years?

c What additional information is needed to be able to predict the long-term population? What is the long-term population in this case?

d Consider the following variations in a fish-population situation modeled by a *NOW-NEXT* formula like the one above.

i. If the initial population doubles, what will happen to the long-term population?

ii. If the annual restocking amount doubles, what will happen to the long-term population?

iii. If the annual population decrease rate doubles, what will happen to the long-term population?

e How would you modify *NEXT* = 0.6*NOW* + 1,500 so that it represents a situation in which the fish population increases annually at a rate of 15%? What effect does such an increase rate have on the long-term population?

Be prepared to explain your ideas to the class.

Check Your Understanding

A hospital patient is given an antibiotic to treat an infection. He is initially given a 30-mg dose and then receives another 10 mg at the end of every six-hour period thereafter. Through natural body metabolism, about 20% of the antibiotic is eliminated from his system every six hours.

a. Estimate the amount of antibiotic in his system after the first six hours and after the second six hours.

b. Write a formula using the words *NOW* and *NEXT* that models this situation.

c. Find the amount of antibiotic in the patient's system after 10 days.

d. Suppose his doctor decides to modify the prescription so that the long-term amount of antibiotic in his system will be about 25 mg. How should the prescription be modified?

Investigation 2 The Power of Notation and Technology

Situations involving sequential change, as in the fish-population problem, are sometimes called **discrete dynamical systems**. A discrete dynamical system is a situation (system) involving change (dynamical) in which the nature of the change is step-by-step (discrete). An important part of analyzing discrete dynamical systems is determining long-term behavior, as you did when you found the long-term fish population or the long-term amount of antibiotic in a patient's system.

Sequential change situations can often be analyzed using *recursion* and *iteration*. Sequential change is step-by-step change. **Recursion** is the method of describing a step in a sequential process in terms of previous steps. **Iteration** is the process of repeating the same procedure or computation over and over again.

Notation and technology can be very helpful when analyzing sequential change situations. As you work on the problems of this investigation, look for answers to these questions:

How can recursion and iteration be used to model and analyze sequential change situations?

How can subscript, function, and spreadsheet notation be used in the modeling process?

How can spreadsheets be used in the modeling process?

1 Think about how recursion and iteration were used in your analysis of the changing fish population in Investigation 1.

a. Describe an instance where you used recursion to model or analyze the fish-population problem.

b. Describe an instance where you used iteration in the fish population problem.

Subscript and Function Notation You can use subscripts and function notation to more compactly write formulas like those in Investigation 1 that use the words *NOW* and *NEXT*. These notations can help you analyze the formulas and the situations they model. Consider the context of a changing fish population from the last investigation. The subscript notation P_n can be used to represent the population after n years. (The notation P_n is read "P sub n.") Thus, P_0 ("P sub 0") is the population after 0 years, that is, the initial population. P_1 is the population after 1 year, P_2 is the population after 2 years, and so on.

Recall that the fish-population problem is based on these three assumptions:

- There are 3,000 fish currently in the pond.
- The population decreases by 20% each year due to natural causes and fish being caught.
- 1,000 fish are added at the end of each year.

2 Find P_0, P_1, and P_2. Compare to the population values from Investigation 1. Sketch a graph of P_n versus n. Include the three values you have just found, for $n = 0$, 1, and 2. Also, indicate the shape of the graph based on all the population values from Investigation 1. A rough sketch by hand is fine.

3 The subscript notation relates closely to the way you have used the words *NOW* and *NEXT* to describe sequential change in many contexts.

a. In the context of a changing population, if P_1 is the population *NOW*, what subscript notation represents the population *NEXT* year?

b. If P_{24} is the population *NEXT* year, what subscript notation represents the population *NOW*?

c. If P_n is the population *NEXT* year, what subscript notation represents the population *NOW*?

d. If P_{n+1} is the population *NEXT* year, what subscript notation represents the population *NOW*?

e. A formula that models the annual change in fish population as described in Investigation 1 is

$$NEXT = 0.8NOW + 1{,}000, \text{ starting at } 3{,}000.$$

i. Rewrite this formula using P_n and P_{n-1} notation.

ii. Rewrite this formula using P_n and P_{n+1} notation.

4 Function notation can often be used interchangeably with subscript notation. For example, both P_0 and $P(0)$ can represent the initial population. The subscript notation is read "P sub 0" and the function notation is read "P of 0." Each notation can be used to represent the population at time 0.

a. Refer to Problem 2 above. In the context of the fish population problem, if $P(n)$ is the population after n years, what are the values of $P(0)$, $P(1)$, and $P(2)$? Calculate $P(3)$.

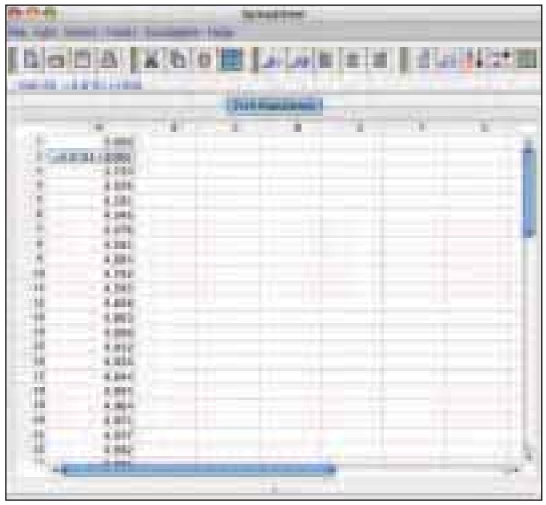
b. Rewrite the *NOW-NEXT* formula from Problem 3 Part e using function notation. Is there more than one way to do this? Explain.

Spreadsheet Analysis You can analyze situations that involve sequential change by using the iteration and recursion capability of a calculator or computer software. In particular, spreadsheet software can be used to analyze these situations.

CPMP-Tools

5 Spreadsheets have their own notation, which is similar to subscript and function notations. Spreadsheets are organized into cells, typically labeled by letters across the top and numbers down the side. If cell A1 contains the fish population *NOW*, and A2 contains the population *NEXT* year, write a formula using A1 and A2 that shows the relationship between *NOW* and *NEXT*. (*Note*: When using formulas that reference spreadsheet cells, a * must be used for multiplication.)

6 Analyze the fish population situation using a spreadsheet, as follows.



a. A formula like that in Problem 5 does not tell you what the population is for any given year; it only shows how the population changes from year to year. To be able to compute the population for any year, you need to know the initial population. Using spreadsheet software available to you, enter the initial population of 3,000 into cell A1.

b. You want A2 to show the population in the next year. So, enter "=" and your expression for A2 from Problem 5 into cell A2. Press Enter or Return. The population for the next year (3,400) should appear in cell A2. If it does not, check with other students and resolve the problem.

c. Now you want A3 to show the population in the next year. So, you need to enter an expression for A3 into the A3 cell. Then to get the next population, you would enter an expression for A4 into the A4 cell. And so on. Repeating this procedure is an example of iteration. The spreadsheet software will do this automatically, using the Fill Down command, as follows.

- Click and hold on cell A2, then drag down to cell A50.
- Then in the Edit menu, choose Fill Down.
- Cells A3 through A50 should now show the population from year to year. If not, check with other students and resolve the problem.

d. Describe the pattern of change you see in the spreadsheet list of population values. Compare to your analysis of the fish-population situation in Investigation 1.

e. You can use the graphing capability of a spreadsheet to help analyze the situation.

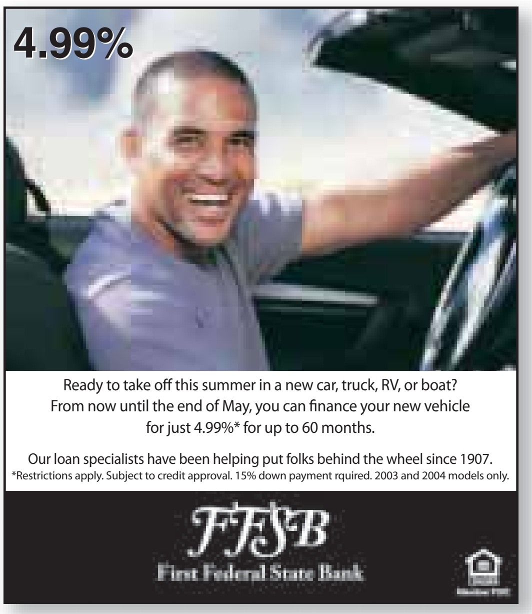
i. Use the spreadsheet software to create a scatterplot of the fish population over time. When there is only one column of data, some spreadsheet software has a default setting that will assign integers beginning with 1 as the independent variable for a scatterplot. If your spreadsheet software does not have this default setting, fill column B with the year numbers that correspond to the populations in column A.

ii. Describe any patterns of change you see in the population graph. Compare to your analysis of the fish-population situation in Investigation 1. Be sure to describe how the long-term population trend shows up in the graph.

Compound Interest You have been using recursion and iteration, along with appropriate notation and technology tools, to analyze a population problem. Another common application of recursion and iteration is compound interest. *Compound interest* is interest that is applied to previous interest as well as to the original amount of money borrowed or invested. The original amount of money is called the *principal*. The more often the interest is applied, that is, the more often it is *compounded*, the faster the total amount of money (principal plus interest) grows. Thus, the interest rate and compounding period are crucial factors in consumer loans, such as car loans and college loans.

In the U.S., the Truth in Lending Act requires that lenders must report interest rates in a standard way, using the Annual Percentage Rate (APR), so that consumers can more easily compare and understand loans. Unfortunately, the APR itself is sometimes interpreted and computed in different ways—see Extensions Task 18. In any case, even with some clarity provided by the APR, you still need to carefully analyze the interest rate on a loan.

For example, it is common to advertise car loans with a stated *annual* interest rate. However, the interest is usually compounded *monthly*, not annually, so careful analysis is required to make sure you know what the loan will really cost. Consider the following car loan ad.



7 The stated interest rate is 4.99%. Given the law about stating the APR, we will assume that this is an annual rate. The ad also implies that the loan will be computed monthly ("for up to 60 months"), so we will also assume that the interest will be compounded monthly. Finally, it is common to give the annual rate without adjusting for the effect of the monthly compounding. To carefully analyze this situation, we need to know the monthly interest rate. Under the assumptions just discussed, the monthly rate is $\frac{1}{12}$ of the annual rate. Explain why. For the annual rate shown in the ad, what is the monthly interest rate?

8 Suppose you borrow \$10,000 to buy a new compact car at the interest rate shown in the ad, with repayment due in 48 monthly payments. You are told that your monthly payment will be \$235. As a wise consumer, you should check to see if this payment amount is correct.

a. Figure out a method for how you can determine the amount you still owe, called the *balance*, after a given month's payment. Then find the balance of the car loan after each of the first three payments. Assume that the first payment is made exactly one month after the contract is signed. Compare your balances to those of other groups, and resolve any differences.

b. Write a formula using the words *NOW* and *NEXT* that models the month-by-month change in the balance of the loan. Write equivalent formulas using subscripts, function notation, and spreadsheet notation. Be sure to specify the initial balance.

c. Now think about whether the payment amount is correct.

- If the monthly payment of \$235 is correct, what should the balance be after the 48th payment? Use the formulas from Part b and your calculator or computer to see if \$235 is the correct payment for this loan.
- If the payment is incorrect, is it too high or too low? How can you tell? Experiment to find the correct monthly payment.

9 Investigate further the loan situation from Problem 8.

a. How much total interest was paid on the \$10,000 loan, using the correct monthly payment?

b. Is the balance of the loan reduced faster at the beginning of the repayment period or at the end? Explain and give evidence to support your answer.

c. Suppose someone offers you a "great deal," whereby you must pay only \$40 per month until the loan is paid off. Describe what happens to the repayment process in this situation. Is this such a great deal after all?

Summarize the Mathematics

Formulas involving recursion, like those you have been using in this investigation, are called recursive formulas. The recursive formulas you have studied have all been of the general form $A_n = rA_{n-1} + b$, with A_0 as the initial value. Consider a wildlife population that is modeled by the following formula and initial value.

$$U_n = 0.4U_{n-1} + 2{,}500,\ U_0 = 11{,}000$$

a What are the values for r and b in the recursive formula for the wildlife population?

b What can you say about the size of the wildlife population and how it is changing?

c Rewrite the "$U_n = \ldots$" formula using the words *NOW* and *NEXT*. Write an equivalent formula using function notation.

d If the initial population is entered into cell **A8** in a spreadsheet and the population next year is entered into cell **A9**, write a formula showing the relationship between **A8** and **A9**.

e The size of the population after 3 years is 4,604. Explain how to find the population after 4 years:

- **i.** without using a calculator or computer.
- **ii.** using the last-answer feature of a calculator or computer.
- **iii.** using a spreadsheet.

f Describe the pattern of population change, including the long-term population trend. Sketch a graph and explain how the graph shows the trend.

g Explain why this is a sequential change situation and how it involves recursion and iteration.

Be prepared to share your responses and thinking with the entire class.

Check Your Understanding

Recall the situation (page 462) involving a hospital patient taking an antibiotic to treat an infection. He was initially given a 30-mg dose, and then he took another 10 mg at the end of every six hours. Through natural body metabolism, about 20% of the antibiotic was eliminated from his system every six hours.

a. Write a recursive formula in the form "$U_n = \ldots$" that models this situation. Write an equivalent formula using function notation.

b. Using spreadsheet software, produce a table and graph of (n, U_n) pairs with n as the input variable.

c. Describe how the amount of antibiotic in the patient's system changes over time, including the long-term change.

On Your Own

Applications

For the following tasks, use spreadsheet software, *CPMP-Tools*, or a graphing calculator as needed to help you solve the problems.

1 Chlorine is used to keep swimming pools safe by controlling certain harmful microorganisms. However, chlorine is a powerful chemical, so just the right amount must be used. Too much chlorine irritates swimmers' eyes and can be hazardous to their health; too little chlorine allows the growth of microorganisms to be uncontrolled, which can be harmful.

A pool manager must measure and add chlorine regularly to keep the level just right. The chlorine is measured in parts per million (ppm) by weight. That is, one ppm of chlorine means that there is one ounce of chlorine for every million ounces of water.

Chlorine dissipates in reaction to bacteria and to the sun at a rate of about 15% of the amount present per day. The optimal concentration of chlorine in a pool is from 1 to 2 ppm, although it is safe to swim when the concentration is as high as 3 ppm.

CPMP-Tools

a. Suppose you have a summer job working at a swimming pool, and one of your responsibilities is to maintain a safe concentration of chlorine in the pool. You are required to add the same amount of chlorine to the pool every day. When you take the job, you find that the concentration is 3 ppm.

How much chlorine (in parts per million) do you need to add each day in order to maintain a long-term optimal concentration? Write a recursive formula that models your optimal chlorine maintenance plan. Describe any assumptions you have made in your analysis.

b. There are three key factors in this problem: the initial concentration, the daily increase in concentration due to the amount you add, and the dissipation rate. Systematically explore changes in each of these three factors and record the corresponding effects on the long-term chlorine concentration in the swimming pool.

c. Suppose the chlorinating pellets you use are 65% active chlorine, by weight. If the pool contains 50,000 gallons of water, and water weighs 8.337 pounds per gallon, how many pounds of chlorine pellets must you add to the pool each day?

2 Retirement is probably not something you are currently concerned about. However, working adults, even very young working adults, should have a financial plan for retirement. If you start saving early and take advantage of compound interest, then you should be in great financial shape by the time you retire. Consider twin sisters with different retirement savings plans.

Plan I: Cora begins a retirement account at age 20. She starts with \$2,000 and then saves \$2,000 per year at 7% interest, compounded annually, for 10 years. (*Compounded annually* means that the interest is compounded every year, once per year at the end of the year.) Then she stops contributing to the account but keeps her savings invested at the same rate.

Plan II: Miranda does not save any money in her twenties. But when she turns 30, she starts with \$2,000 and then saves \$2,000 per year at 7% interest, compounded annually, for 35 years.

Both sisters retire at age 65. Who do you think will have more retirement savings at age 65? Test your conjecture by determining the amount of money saved by each sister at age 65.

3 Every ten years, the United States Census Bureau conducts a complete census of the nation's population. In 2000, the census report stated that there were about 281 million residents in the United States and its territories. The population changes extensively between census reports, but it is too expensive to conduct the census more often. Annual changes can be determined using estimates like the following.

- Births will equal about 1.5% of the total population each year.
- Deaths will equal about 0.9% of the total population each year.
- Immigrants from other countries will add about 0.8 million people each year. (Source: 2001 World Population Data Sheet, Population Reference Bureau, www.prb.org.)

a. Using the statistics above, what population is estimated for the United States in 2001? In the year 2010?

b. Write a formula using the words *NOW* and *NEXT* that represents this situation.

c. Write a recursive formula that represents this situation. Specify the initial value.

d. Produce a table and a graph that show the population estimates through the year 2020. Describe the expected long-term trend in population change over time.

e. Describe some hypothetical birth and death rates that would result in a population that levels off over time. Represent this situation with a recursive formula, a table, and a graph.

4 Commercial hunting of whales is controlled to prevent the extinction of some species. Because of the danger of extinction, scientists conduct counts of whales to monitor their population changes. A status report on the bowhead whales of Alaska estimated that the 1993 population of these whales was between 6,900 and 9,200 and that the difference between births and deaths yielded an annual growth rate of about 3.1%. No hunting of bowhead whales is allowed, except that Alaskan Inuit are allowed to take, or harvest, about 50 bowhead whales each year for their livelihood. (Source: nmml.afsc.noaa.gov/CetaceanAssessment/bowhead/bmsos.htm)

a. Use 1993 as the initial year. The initial population is a range of values, rather than a single value. That is, the range of initial population values is 6,900 to 9,200. Think about the *range* of population values for later years. Using the given information about annual growth rate and harvesting by Alaskan Inuit, what is the range of population values in 1994?

b. Let LOW(n) be the lower population value in the range of values in year n.

- **i.** What is the initial value, that is, what is LOW(0)? What is LOW(1)? Write a recursive formula for the growth of the lower population values in the range of population values.
- **ii.** Similarly, let HIGH(n) be the higher population value in the range of values in year n. Write a recursive formula for the growth of the higher population values in the range of population values, and state the initial value.

c. Put the formulas in Part b together to write a recursive formula for this situation, by completing the following work. Fill in the missing steps and provide the requested explanations.

Let $R(n)$ represent the population range in year n.

(1)	So, $R(0) = 6{,}900$ to $9{,}200$.	Explain.
(2)	$R(1) = 7{,}063.9$ to $9{,}435.2$.	Explain.
(3)	$R(2) =$	Fill in the range for $R(2)$.
(4)	$R(n) =$	Fill in the range for $R(n)$, using the formulas from Part b. Include the initial values for both the lower and higher population values.

d. Sketch a graphical representation for $R(n)$, by drawing vertical line segments for the range corresponding to each year n, with values of n on the horizontal axis and population values on the vertical axis. Use at least 4 years. Elaborate on this graphical representation by sketching graphs for n vs LOW(n) and n vs HIGH(n) on the same set of axes. Describe some patterns in the ranges $R(n)$. (See Lesson 2, Extensions Task 25 for further investigation of this situation.)

e. So far in this task, population values have been growing. Suppose that, because of some natural disaster, the initial bowhead whale population is reduced to 1,500, but growth rate and number harvested by the Inuit stay the same. Under these conditions, what happens to the long-term population?

5 Money grows when it is kept in an interest-bearing savings account. Recursive formulas can be used to analyze how the money grows due to compound interest. For this task, assume that you deposit money into a savings account and make no withdrawals.

a. Suppose you deposit \$100 in a savings account that pays 4% interest, compounded annually. (*Compounded annually* means that the interest is compounded every year, once per year at the end of the year.) Write a *NOW-NEXT* formula and a recursive formula that show how the amount of money in your account changes from year to year. Find the amount of money in the account after 10 years.

b. Most savings accounts pay interest that compounds more often than annually. Suppose that you make an initial deposit of \$100 into an account that pays 4% annual interest, compounded monthly.

i. Write a recursive formula that models the month-by-month change in the amount of money in your account.

ii. How much money is in the account after 1 month? After 2 months? After 2 years?

iii. How much money is in the account after 10 years? Compare your answer to the answer you got in Part a. Which kind of interest is a better deal, compounding annually or compounding monthly?

c. Suppose that, in addition to the initial \$100, you deposit another \$50 at the end of every year, and the interest rate is 4% compounded annually.

i. Write a recursive formula that models this situation.

ii. How much money is in the account after 10 years?

iii. Describe the pattern of growth of the money in the savings account.

6 In this lesson, you have investigated recursive formulas of the form $A_n = rA_{n-1} + b$. Different values for r and b yield models for different situations. Consider some of the possibilities by completing a table like the one on the next page. For each cell in the table, do the following.

- Describe a situation that could be modeled by a recursive formula with the designated values of r and b. (You may use examples from the lesson if you wish.)
- Write the recursive formula.
- Describe the long-term trend.

	$0 < r < 1$	$r > 1$
$b < 0$	A whale population is decreasing by 5% per year due to a death rate higher than the birthrate, and 50 whales are harvested each year. $A_n = 0.95A_{n-1} - 50$ Long-term trend: Extinction	
$b > 0$		

Connections

7 In Investigation 2, you found the following recursive formula to model a fish population.

Formula I $P_n = 0.8P_{n-1} + 1{,}000$, with $P_0 = 3{,}000$

Here is another formula that represents this situation.

Formula II $P_n = -2{,}000(0.8^n) + 5{,}000$

a. You can provide an algebraic argument for why the two formulas above are equivalent (see Connections Task 11 in Lesson 2 and Extensions Task 18 in Lesson 3). For now, give an informal argument using graphs, as follows.

i. Use geometric transformations to describe how the graph of the function indicated by Formula II is related to the graph of the exponential function $f(n) = 0.8^n$.

ii. Compare the graph of the function given by Formula II to the graph you generated in Investigation 2 using Formula I. Describe how the graphs are similar.

iii. Why do you think it makes sense that these two formulas could be equivalent models of the fish-population problem?

b. Verify that Formula II gives the same values for P_1 and P_5 as those found using the recursive formula.

c. Verify that Formula II yields an initial population of 3,000.

d. Use Formula II to verify the long-term population you found in Problem 2 of Investigation 1 (page 466). Explain your thinking. How is the long-term population revealed in the symbolic form of this formula?

8 In Investigation 2, there was a brief discussion of Annual Percentage Rate (APR). The APR is meant to be a standard way to state interest rates on loans so that different loans can be compared and understood more easily. A related method of stating interest rates is used for investments, like when you deposit money into a savings account.

The **Annual Percentage Yield** (**APY**) expresses an interest rate on an investment in a way that takes into account the effect of compounding the interest. The APY is different, and usually higher, than a "nominal" annual rate. For example, consider the table below showing interest rates for the Los Alamos National Bank in New Mexico.

Savings Account Interest Rates

as of April 3, 2008

TYPE	INTEREST RATE	APY*	MINIMUM BALANCE**	TIER BALANCE
Regular Savings Account	1.000%	1.010%	None	None
Campus Savings Account	1.000%	1.010%	None	None
Universal Savings Tier 1	1.000%	1.010%	None	$0.00 - $49,999.00
Universal Savings Tier 2	1.750%	1.770%	None	$50,000.00 - $99,999.00
Universal Savings Tier 3	2.500%	2.530%	None	$100,000.00 - $999,999.00
Universal Savings Tier 4	2.750%	2.790%	None	$1,000,000.00 and above
Money Market Tier 1	0.500%	0.500%	$1,000.00	$0.00 - $19,999.00
Money Market Tier 2	1.750%	1.770%	$1,000.00	$20,000.00 - $49,999.00
Money Market Tier 3	2.250%	2.280%	$1,000.00	$50,000.00 - $99,999.00
Money Market Tier 4	2.500%	2.530%	$1,000.00	$100,000.00 and above

The above Interest Rates are subject to change without prior notice.

*Annual Percentage Yield
**Minimum balance required to avoid periodic service charge.

Source: www.lanb.com/rates/savings.asp, retrieved April 3, 2008

This bank pays interest compounded monthly. The rates shown in the column labeled "INTEREST RATE" are sometimes called *nominal interest rates*. These rates are annual rates, and they do not take into account the effect of compounding. The APY is the nominal annual rate adjusted for the effect of monthly compounding. Here is how it works.

a. Let's think big and consider Universal Savings Tier 3. The stated nominal interest rate is 2.500%. This is an annual rate.

  i. What is the corresponding monthly rate?

  ii. Suppose you invest $1 at this monthly rate and you compound the interest for 12 months. How much will you have at the end of the year?

b. 2.500% is an annual interest rate that does *not* take into account the monthly compounding. Explain why the number you determined in Part a can be thought of as an *annual growth rate* that *does* take the monthly compounding into account.

c. Subtract 1 from the number you determined in Part a. Compare the result to the APY for Universal Savings Tier 3. Describe similarities and differences.

d. The formula for computing APY is

$$APY = \left(1 + \frac{r}{n}\right)^n - 1,$$

where r is the nominal annual rate, expressed as a decimal, and n is the number of compound periods in a year.

i. Explain how each part of this formula relates to the numbers and computations in Parts a–c.

ii. Use this formula to compute the APY for another type of savings account in the table on the previous page.

9 You can represent recursive formulas with subscript notation or function notation. For example, in the fish-population problem, you can think of the population P as a function of the number of years n and you can write P_n or $P(n)$ for the population after n years. In the case of the fish-population problem, what are the practical domain and practical range of the function P?

10 Think about how matrix multiplication can be used to calculate successive values generated by recursive formulas of the form $A_n = rA_{n-1} + b$. For example, consider the recursive formula for the original fish-population situation at the beginning of this lesson: $P_n = 0.8P_{n-1} + 1{,}000$, with $P_0 = 3{,}000$.

a. A first attempt at a matrix multiplication that would be equivalent to evaluating the recursive formula might use the following matrices.

$$A = \begin{bmatrix} 3{,}000 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.8 \\ 1{,}000 \end{bmatrix}$$

Compute AB and compare it to P_1.

b. AB is just a 1×1 matrix and so is not much good for repeated multiplication. Thus, you cannot use repeated multiplication of these matrices to successively evaluate the recursive formula. A better try might be to use the following matrices.

$$A = \begin{bmatrix} 3{,}000 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0.8 & 0 \\ 1{,}000 & 1 \end{bmatrix}$$

Compute AC and compare it to P_1. What matrix multiplication would you use next to find P_2? To find P_3?

c. There is just one finishing touch needed. Modify the matrices so that the multiplication computes not only the successive values but also the number of the year. Consider the following matrices.

$$A = \begin{bmatrix} 0 & 3{,}000 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0 \\ 1 & 1{,}000 & 1 \end{bmatrix}$$

Compute AD, AD^2, AD^3, AD^4, and AD^5. Compare the results to P_1 through P_5.

d. Use matrix multiplication to generate three successive values of the recursive formula $P_n = 1.04P_{n-1} - 350$, with $P_0 = 6{,}700$. Do so in a way that also generates the successive values of n.

e. Explain why this matrix multiplication method works to find successive values of a recursive formula.

Reflections

11 When using some graphing calculators, you have the option of graphing in "connected" mode or "dot" mode. Find out what the difference is between these two modes of graphing. Which do you think is more appropriate for the graphing you have been doing in this lesson? Why?

12 Consider again the fish population change modeled by $NEXT = 0.8NOW + 1{,}000$, from Investigation 1. You used several different formulas to represent this fish population, for example, a formula using the words *NOW* and *NEXT*, a formula with subscripts using P_n and P_{n-1}, a formula with subscripts using P_n and P_{n+1}, a formula with function notation using $A(n)$ and $A(n + 1)$, and a formula using spreadsheet notation.

Suppose that one of your classmates has not completed this task. Write a paragraph to the classmate explaining how all these formulas accurately represent the changing fish population.

13 In this lesson, you used recursive formulas to model sequential change in several situations, such as population growth, drug concentration, chlorine concentration, and money saved or owed. Describe one result of your investigations that was particularly interesting or surprising.

14 Recursive formulas are also sometimes called *difference equations*. What difference would be of interest in studying a recursive formula?

Extensions

15 These days, almost every state has a lottery, and the jackpots are often quite large. But are they really as large as they seem? Suppose you win \$500,000 in a lottery (which you will, of course, donate to the mathematics department at your school). Typically, these large jackpots are not paid immediately as a lump sum. Instead, they are paid over time. For example, suppose you receive your \$500,000 over 20 years at \$25,000 per year. To accurately analyze how much you have *really* won in this situation, you need to include the effect of compound interest.

a. Suppose you deposit \$500,000 in a bank account paying 3.5% annual interest, and you withdraw \$25,000 at the end of every year. Write a recursive formula that models this situation, and calculate how much money will be in the account after 20 years.

b. Experiment to find an initial deposit that will yield a balance of \$0 after 20 years.

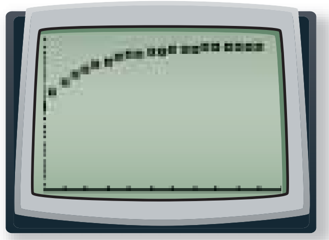
c. The result from Part b is called the **present value** of your lottery winning. The present value is the lump sum amount that, if deposited now at 3.5% annual interest, would generate payments of $25,000 per year for 20 years. The present value is what your lottery winning is really worth, taking into account the reality of compound interest. So in this situation, how much is a $500,000 jackpot paid over 20 years really worth?

d. Find the actual value, that is, the present value, of a lottery winning of $1,000,000 that is paid at $50,000 per year for 20 years, if you can invest money at 5% annual interest.

e. What other factors besides compound interest do you think could affect the present value?

16 In this lesson, you modeled a fish population with the recursive formula $P_n = 0.8P_{n-1} + 1{,}000$, with $P_0 = 3{,}000$. You can get another modeling formula for this situation by fitting a curve to the population data and finding the equation for that curve. Use the following steps to carry out this plan.

a. Use the formula $P_n = 0.8P_{n-1} + 1{,}000$, with $P_0 = 3{,}000$, to generate the sequence of population figures for 20 years. Put these figures into a data list, say L2, on your calculator or computer. Generate another list, say L1, that contains the sequence of years 0 through 20. Produce a scatterplot of L1 and L2.



b. Modify your scatterplot by transforming the data in the lists so that the plot matches one of the standard regression models of your calculator or computer software. Find a regression equation that fits the transformed data.

c. Now, transform your regression equation so that it fits the original data.

d. Test your equation. Does the graph fit the original scatterplot? Compare the function value for $n = 20$ to what you get when you find P_{20} using the recursive formula $P_n = 0.8P_{n-1} + 1{,}000$, with $P_0 = 3{,}000$.

17 The Towers of Hanoi is a mathematical game featured in an old story about the end of the world. As the story goes, monks in a secluded temple are working on this game; and when they are finished, the world will end! The people of the world would like to know how long it will take the monks to finish the game.

The game is played with 3 pegs mounted on a board and 64 golden disks of successively larger sizes with holes in the center. (Commercial games such as the example on the previous page have many fewer disks.) The game begins with all 64 disks stacked on one peg, from largest to smallest. The goal of the game is to move all 64 disks onto another peg, subject to these rules: (1) Only one disk may be moved at a time. (2) You are not allowed to put a larger disk on top of a smaller disk. (3) Disks may be placed temporarily on any peg, including the one on which the disks were originally stacked. (4) Eventually, the disks must be stacked from largest to smallest on a new peg.

a. You may not have 64 golden disks to play this game. But to get an idea of how it works, you can play with some homemade disks and pegs. Cut out four successively larger squares of paper to use as disks (or use different size coins for the disks). Put three large dots on a piece of paper to use as pegs mounted on a board. Play the game several times, first with just one disk stacked on the starting peg, then with two disks, then three, and so on. As you play each game, keep track of the *fewest* number of moves it takes to finish the game. Make a table listing the number of disks in the game and the fewest number of moves it takes to finish the game.

b. By thinking about strategies for how to play the game with more and more disks and by looking for patterns in the table from Part a, find a recursive formula and any other formula you can for the fewest number of moves needed to finish the game with n disks.

c. What is the fewest number of moves needed to finish a game with 64 disks? If the monks in the story move one disk every second and work nonstop, should we worry about the world ending soon? Explain.

18 The Annual Percentage Rate (APR), which you considered in Investigation 1, is not interpreted consistently in all ads and descriptions on the Internet. This may be because the law on which the APR is based is long and complicated. The APR comes from the U.S. Federal Truth in Lending Act, which is contained in Title I of the Consumer Credit Protection Act from May 29, 1968. This Act is implemented by the Truth in Lending Act Regulation, known as Regulation Z, as amended on December 20, 2001. At one place in these laws, it is stated that:

> "The annual percentage rate [APR] … shall be determined … as that annual percentage rate which will yield a sum equal to the amount of the finance charge when it is applied to the unpaid balances of the amount financed … ." Furthermore, "the amount of the finance charge shall be determined as the sum of all charges … which include interest, [and other charges]." (Sections 106 and 107 of the Truth in Lending Act)

Elsewhere in the law it is stated that:

"The annual percentage rate [APR] shall be the nominal annual percentage rate determined by multiplying the unit-period rate by the number of unit-periods in a year." (Appendix J of Regulation Z)

a. Suppose a lender advertises a loan with an interest rate of 0.8% per month compounded monthly.

i. Using the first statement from the Truth in Lending Act, which implies that the APR must be stated as an annual rate that takes into account compounding interest, what annual interest rate might the lender state?

ii. Using the second statement above, from Regulation Z, what annual rate might the lender state?

b. Explain how the different interpretations of the APR that may be possible based on the two statements above can be described in terms of the difference between growth determined by an exponential function and growth determined by a linear function.

19 Suppose you have a recursive formula that can be used to compute the next term in a sequence from the current term. Think about how you could compute every second term or perhaps every tenth term. (For example, you might want to predict the population every decade instead of every year.)

a. Suppose $NEXT = 3NOW$ and the starting value is 1. List the first 10 terms in the sequence generated by this recursive formula and initial value. Starting with 1, list every second term within this list. Write a recursive formula that describes the list of every second term.

b. Think about every tenth term in the original sequence in Part a, that is, the sequence determined by $NEXT = 3NOW$, starting at 1. Write a new recursive formula that would generate every tenth term in the sequence.

c. Consider the sequence of numbers generated by $A_n = 3A_{n-1} + 2$, with $A_0 = 1$. List the first 11 terms of this sequence. Think about new recursive formulas that will generate every second term or every tenth term of this sequence.

i. Let $\{S_n\}$ be the sequence of every second term, starting at 1. Explain each step in the following argument to find a recursive formula for S_n.

$$
\begin{aligned}
S_0 &= A_0 = 1 && (1)\\
S_1 &= A_2 = 17 && (2)\\
S_2 &= A_4 = 161 && (3)\\
&\vdots\\
S_n &= A_{2n} && (4)\\
&= 3A_{2n-1} + 2 && (5)\\
&= 3(3A_{2n-2} + 2) + 2 && (6)\\
&= 3^2A_{2n-2} + 3(2) + 2 && (7)\\
&= 3^2A_{2(n-1)} + 3(2) + 2 && (8)\\
&= 3^2S_{n-1} + 3(2) + 2 && (9)
\end{aligned}
$$

Thus, $S_n = 3^2S_{n-1} + 3(2) + 2$.

ii. Let $\{T_n\}$ be the sequence of every tenth term, starting at 1. Explain each step in the following argument to find a recursive formula for T_n.

$$
\begin{aligned}
T_n &= A_{10n} && (1)\\
&= 3A_{10n-1} + 2 && (2)\\
&= 3(3A_{10n-2} + 2) + 2 && (3)\\
&= 3^2A_{10n-2} + 3(2) + 2 && (4)\\
&= 3^2(3A_{10n-3} + 2) + 3(2) + 2 && (5)\\
&= 3^3A_{10n-3} + 3^2(2) + 3(2) + 2 && (6)\\
&\vdots\\
&= 3^{10}A_{10n-10} + 3^9(2) + 3^8(2) + \cdots + 3(2) + 2 && (7)\\
&= 3^{10}A_{10(n-1)} + 3^9(2) + 3^8(2) + \cdots + 3(2) + 2 && (8)\\
&= 3^{10}T_{n-1} + 3^9(2) + 3^8(2) + \cdots + 3(2) + 2 && (9)
\end{aligned}
$$

Thus, $T_n = 3^{10}T_{n-1} + 3^9(2) + 3^8(2) + \cdots + 3(2) + 2$. Use this formula to compute T_1. Compare to A_{10}, which you computed at the beginning of Part c on the previous page.

20 So far, you have used recursive formulas to predict future values in a sequential change situation. This is a natural use of recursive formulas since they typically show how to get from one term in a sequence to the next. Think about reversing the perspective and computing past terms.

a. Suppose $NEXT = 2NOW$, and one of the terms in the sequence is 460.8. What is the previous term? Describe how you found the previous term.

b. Suppose a changing population is modeled by this recursive formula.

$P_n = 0.8P_{n-1} + 500$, where P_n is the population at year n

If the population predicted by this formula in a given year is 1,460, what is the population the previous year?

c. The recursive formula in Part b can be used to compute the population in future years. Construct a new recursive formula that you could use to compute the population in past years. Use the new recursive formula to compute the population 10 years ago if the current population is 2,341.

Review

21 A study of milk samples from 167,460 Holstein cows measured the percent of fat in each milk sample. The data was normally distributed with mean milk fat of 3.59% and a standard deviation of 0.71%.

a. Approximately how many of the samples had milk fat between 2.17% and 5.01%?

b. Approximately how many of the samples had less than 4.3% milk fat?

c. Approximately what percentage of the samples had more than 5.01% milk fat?

22 Perform each indicated operation. Write the polynomial in standard form.

a. $(3x^3 - 5x + 14) - (x^3 + 4x^2 - 6x)$

b. $(6 - x^3)^2(x + 1)$

c. $8(-x^2 + 7) + 5(x^4 - 5x^2 + 10)$

d. $(x^3 + 5x) - (7x^3 - 3x) + (4x^2 + 8x)$

23 Determine whether each data pattern is linear, exponential, or quadratic. Then find a function rule that matches each pattern.

a.

x	−3	−2	−1	0	1	2	3
y	3	9	27	81	243	729	2,187

b.

x	−4	−3	−2	0	2	4	7
y	1	5	9	17	25	33	45

c.

x	−2	−1	0	1	2	3	4
y	4,800	2,400	1,200	600	300	150	75

d.

x	−3	−2	−1	0	1	2	3
y	20	12	6	2	0	0	2

24 Simplify each algebraic fraction as much as possible. Then determine if the two expressions are undefined for the same values.

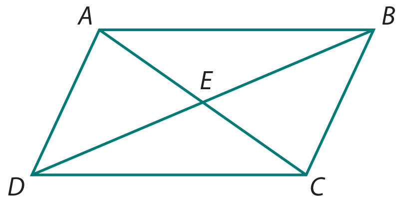
a. $\dfrac{12x^2 + 4x}{3x}$

b. $\dfrac{2x}{6x - 8}$

c. $\dfrac{x^2 - 9}{x + 3}$

d. $\dfrac{x^2 + 2x - 8}{x^2 - 7x + 10}$

25 Determine if each set of conditions will guarantee that quadrilateral *ABCD* is a parallelogram. If not, draw a counterexample. If so, provide a proof.



a. $\overline{CE} \cong \overline{EA}$

b. $\triangle AED \cong \triangle CEB$

c. $\overline{AC} \perp \overline{DB}$

26 Solve each exponential equation.

a. $10^x = 0.01$

b. $5(10^x) = 5{,}000$

c. $25(10^x) = 1{,}300$

d. $12(3^x) = 324$

LESSON

2

A Recursive View of Functions

In the previous lesson, you investigated a variety of situations that were modeled by recursive formulas, including population growth, consumer loans, and medicine dosage. In each of these situations, you examined a sequence of numbers and looked for patterns and formulas to represent the situation. In this lesson, you will take a closer look at sequences of numbers. As you do so, you will extend your understanding of recursion and iteration. In the process, you will revisit three important function families: linear, exponential, and polynomial. To begin, consider the exciting but dangerous sport of sky diving.

Think About This Situation

Imagine a sky diver jumping from a plane at a height of about 5,000 feet. Because of Earth's gravity and ignoring wind resistance, the sky diver will fall 16 feet in the first second. Thereafter, until the parachute opens, the distance fallen during each second will be 32 feet more than the distance fallen during the previous second.

a Think about the pattern of change in the distance the sky diver falls *during* each second. How would you describe the number of feet fallen during each second:

- **i.** using a recursive formula?
- **ii.** as a function of the number of seconds n?

b Now consider the pattern of change in the *total* distance fallen *after* a given number of seconds.

- **i.** How far has the sky diver fallen after each of the first four seconds?
- **ii.** Describe any patterns you see in this sequence of numbers. Compare this sequence to the sequence of distances fallen during each second.

c Think about a possible formula for the total distance fallen.

- **i.** How could the total distance fallen be computed from the sequence of numbers in Part a?
- **ii.** If you found a function formula for the total distance fallen after n seconds, what type of function do you think it will be?
- **iii.** How might a sky diver use a formula for total distance fallen in planning a jump?

In this lesson, you will learn about arithmetic and geometric sequences and series, and finite differences tables. In the process, you will use recursion and iteration to revisit linear, exponential, and polynomial functions.

Investigation 1 Arithmetic and Geometric Sequences

Sequences of numbers occur in many situations. Certain sequences are particularly common and important, such as *arithmetic* and *geometric* sequences. You will learn about these sequences in this investigation. As you work on the problems of this investigation, look for answers to these questions:

What are formulas for arithmetic and geometric sequences?

What functions correspond to arithmetic and geometric sequences?

Arithmetic Sequences As you can imagine, sky diving requires considerable training and careful advance preparation. Although the sport of bungee jumping may require little or no training, it also requires careful preparation to ensure the safety of the jumper. In Course 1, you may have explored the relationship between jumper weight and bungee cord length by conducting an experiment with rubber bands and weights.

1 In one such experiment, for each ounce of weight added to a 3-inch rubber band, the rubber band stretched about $\frac{1}{2}$ inch.

a. Describe the relationship between weight added and stretched rubber band length. What type of function represents this relationship?

b. Complete a copy of the table below.

Bungee Experiment

Weight (in ounces)	0	1	2	3	4	5	…	50	…	99	…	n
Length (in inches)							…		…		…	

c. The sequence of numbers that you get for the lengths is called an *arithmetic sequence*. An **arithmetic sequence** of numbers is one in which you add the same constant to each number in the sequence to get the next number. An arithmetic sequence models **arithmetic growth**. Explain why the sequence of stretched rubber band lengths is an arithmetic sequence.

d. Which, if any, of the sequences in the Think About This Situation is an arithmetic sequence? Why?

2 There are several different formulas that can be used to represent the weight-length relationship.

a. Write a formula using the words *NOW* and *NEXT* that shows how the length changes as weight is added.

b. Write a formula for this situation that looks like

$$L_n = (\text{expression involving } L_{n-1}).$$

This is called a *recursive formula for the sequence*.

c. Use the recursive formula to predict the rubber band length when 15 ounces of weight are attached.

d. Write a formula for the weight-length situation that looks like

$$L_n = \text{(expression involving } n\text{, but not } L_{n-1}\text{)}.$$

This form can be called a *function formula for the sequence* of lengths.

i. Explain why it makes sense to call this form a function formula.

ii. Rewrite the formula using function notation.

e. Use the function formula in Part d to predict the rubber band length when 15 ounces of weight are attached.

f. Describe the difference between the processes of using the recursive formula and using the function formula to compute the rubber band length when 15 ounces of weight are attached.

3 Cell phones, pagers, and telephone calling cards are convenient ways to stay in contact with friends and family. The details of phone cards can be quite complicated, even though the prices and plans may look simple. For example, there might be connection fees, payphone surcharges, and different schemes for rounding the number of minutes used. Suppose you have a phone card with a rate of 2¢ per minute and no other fees or charges, except for a payphone charge of 89¢ per call. When the minutes used are not an integer, the value is rounded to the next minute.

a. Suppose you make a call from a payphone using this phone card. Determine recursive and function formulas for the minute-by-minute sequence of phone card charges for this call.

b. Compare these formulas to those you found for the rubber band experiment in Problem 2. How are they similar? How are they different?

4 Now think about possible connections between arithmetic sequences and functions you have studied.

a. In each case, describe the shape of the graph of the function formula.

b. What is the slope of each graph? How does the slope appear in each of the recursive and function formulas you have been examining?

5 Suppose t_0 is the initial term of a *general* arithmetic sequence for which you add d to each term of the sequence to get the next term of the sequence.

a. Write the first five terms of this sequence. Then find recursive and function formulas for the sequence. Compare your formulas to those of other students. Resolve any differences.

b. LaToya explained her function formula as follows.

> This sequence starts with t_0, and then to get to t_n you add d, n times. Thus, the general formula is $t_n = t_0 + dn$.

Explain how LaToya's explanation matches her formula.

c. How would you interpret t_0, d, and n in the context of the rubber band experiment?

d. The constant d is sometimes called the **common difference** between terms. Explain why it makes sense to call d the common difference.

e. Suppose an arithmetic sequence $t_0, t_1, t_2, \ldots$ begins with $t_0 = 84$ and has a common difference of -6. Find function and recursive formulas for this sequence and then find the term t_{87}.

Geometric Sequences Now consider a different type of sequence. As you investigate this new type of sequence, think about similarities to and differences from arithmetic sequences.

6 Consider the growth sequence of bacteria cells if a cut by a rusty nail puts 25 bacteria cells into a wound and then the number of bacteria doubles every quarter-hour.

a. Use words, graphs, tables, and algebraic rules to describe the relationship between the number of quarter-hours and the number of bacteria. What type of function represents this relationship?

b. Complete a copy of the table below for this situation.

Bacterial Growth

Number of Quarter-Hours	0	1	2	3	4	5	…	50	…	99	…	n
Bacteria Count							…		…		…	

c. The sequence of numbers that you get for the bacteria count is called a *geometric sequence*. A **geometric sequence** of numbers is one in which each number in the sequence is multiplied by a constant to get the next number. Explain why the sequence of bacteria counts is a geometric sequence.

d. Which, if any, of the sequences in the Think About This Situation (page 482) is a geometric sequence? Why?

7 If B_n represents the bacteria count after n quarter-hours, then there are several different formulas that can be used to model this situation.

a. Write a formula using the words *NOW* and *NEXT* that shows how the bacteria count increases as time passes.

b. Write a recursive formula beginning "$B_n = \ldots$" for the sequence of bacteria counts. Use the recursive formula to predict the bacteria count after 18 quarter-hours.

c. Write a function formula beginning "$B_n = \ldots$" for the sequence of bacteria counts. Use the function formula to predict the bacteria count after 18 quarter-hours.

d. Describe the difference between the processes of using the recursive formula and using the function formula to compute the bacteria count after 18 quarter-hours.

8 One of the simplest fractal patterns is the Sierpinski carpet. Starting with a solid square "carpet" one meter on a side, smaller and smaller squares are cut out of the carpet. The first two stages in forming the carpet are shown below.

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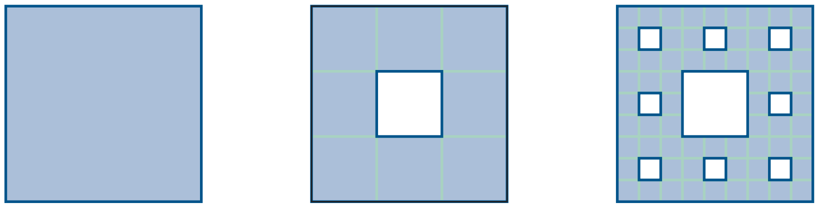


Stage 0 Stage 1 Stage 2

a. Find recursive and function formulas for the sequence of carpet area that remains at each stage.

b. Compare your formulas to those you found for the bacteria present after a cut by a rusty nail, in Problem 7. How are they similar? How are they different?

9 Think about connections between geometric sequences and functions you have studied.

a. What type of function is represented by each of the function formulas in Problems 7 and 8?

b. The growth (or decay) modeled by a geometric sequence is called **geometric growth**. It is also called **exponential growth**. Explain why *exponential growth* is a reasonable way to describe geometric sequences and geometric growth.

10 Suppose t_0 is the initial term of a *general* geometric sequence for which each term of the sequence is multiplied by r ($r \neq 1$) to get the next term of the sequence.

a. Write the first five terms of this sequence and then find recursive and function formulas for the sequence. Compare your formulas to those of other students. Resolve any differences.

b. Eva explained her function formula as follows.

> I started the sequence with t_0, then I multiplied by r each time to get the next terms. To get to t_n, I multiplied by r, n times. Thus, $t_n = t_0 r^n$.

Explain how Eva's explanation matches her formula.

c. What are t_0, r, and n in the situation involving bacterial growth?

d. The constant r is sometimes called the **common ratio** of terms. Explain why it makes sense to call r the common ratio.

e. Suppose a geometric sequence $t_0, t_1, t_2, \ldots$ begins with $t_0 = -4$ and has a common ratio of 3. Find function and recursive formulas for this sequence and then find the term t_{17}.

Different Starting Points The sequences you have studied in this investigation have started with $n = 0$. For example, the initial term for the sequence of rubber band lengths was L_0, and the initial term for the sequence of bacteria counts was B_0. It is possible to have different starting subscripts. Consider what happens to the recursive and function formulas when the starting subscript is 1 instead of 0.

11 In the Think About This Situation, you considered the distance fallen by a sky diver. The table below shows the distance fallen in a free fall by a sky diver, assuming no wind resistance, during each of the first nine seconds of a jump.

Sky Diving Free Fall

Time n (in seconds)	1	2	3	4	5	6	7	8	9
Distance Fallen during Second n, D_n (in feet)	16	48	80	112	144	176	208	240	272

a. Is D_n an arithmetic sequence, geometric sequence, or neither? Explain.

b. What is the starting value of n for this sequence?

c. What is a recursive formula for this sequence?

d. In the Think About This Situation, you may have figured out that a function formula for this sequence is

$$D_n = 16 + 32(n - 1).$$

Check this formula for a few values of n. Explain this formula by modifying LaToya's explanation in Problem 5 on page 485, so that the explanation fits this formula.

12 In Problem 5, you found that a function formula for an arithmetic sequence with initial term t_0 and common difference d is

$$t_n = t_0 + dn.$$

Using the form "$t_n = \ldots$," write a similar function formula if the initial term is t_1, instead of t_0.

13 Consider this geometric sequence: 3, 6, 12, 24, 48, … .

a. What is the common ratio r?

b. Call the general term of this sequence A_n. Suppose that the starting subscript for the sequence is 0. Thus, the initial term is $A_0 = 3$. What is a recursive formula for this sequence? What is a function formula for this sequence? (Write both formulas in the form "$A_n = \ldots$.")

c. Suppose the starting subscript is 1. Thus, $A_1 = 3$. What is a recursive formula for the sequence in this case? What is a function formula? (Write both formulas in the form "$A_n = \ldots$.")

d. Compare the formulas in Parts b and c. Explain similarities and differences.

14 Working individually, write the first five terms for three different sequences. One sequence should be arithmetic, one should be geometric, and the third should be neither.

a. Challenge a classmate to correctly identify your sequence type.

b. Describe how you can tell by inspection whether a sequence is arithmetic, geometric, or neither.

c. For the two sequences you constructed that are arithmetic or geometric, find a recursive and a function formula. For one sequence, choose the starting subscript to be 0. For the other, choose the starting subscript to be 1.

Summarize the Mathematics

In this investigation, you studied two important patterns of growth: arithmetic growth, modeled by arithmetic sequences, and geometric growth, modeled by geometric sequences. You also investigated recursive and function formulas for sequences and made connections to linear and exponential functions.

a Consider arithmetic and geometric sequences.

- **i.** How are arithmetic and geometric sequences different? How are they similar?
- **ii.** Describe one real-world situation different from those you studied in this investigation that could be modeled by an arithmetic sequence. Do the same for a geometric sequence.
- **iii.** What is the connection between arithmetic and geometric sequences and linear and exponential functions?

b Consider recursive and function formulas for sequences.

- **i.** Describe the difference between a recursive formula and a function formula for a sequence. What information do you need to know to find each type of formula?
- **ii.** What is one advantage and one disadvantage of a recursive formula for a sequence? What is one advantage and one disadvantage of a function formula?

c Explain why changing the starting subscript for a sequence has no effect on a recursive formula. Explain why this will cause a change in a function formula.

d In the first lesson of this unit, you investigated situations that could be modeled by recursive formulas of the form $A_n = rA_{n-1} + b$. You can think of these formulas as *combined recursive formulas*. What is the connection between such combined recursive formulas and recursive formulas for arithmetic and geometric sequences? Why does it make sense to call these formulas "combined recursive formulas?"

e You have represented sequences using recursive and function formulas. A function formula for a sequence is also called an **explicit formula** or a **closed-form formula**. Why do you think the terms "explicit" or "closed-form" are used?

Be prepared to explain your ideas to the class.

Check Your Understanding

For each of the sequences below, do the following.

- Complete a copy of the table.
- State whether the sequence is arithmetic, geometric, or neither.
- For those sequences that are arithmetic or geometric, find a recursive formula and a function formula.
- If a sequence is neither arithmetic nor geometric, find whatever formula you can that describes the sequence.

a.

n	0	1	2	3	4	5	6	…	10	…	100	…
A_n	2	6	18	54	162							

b.

n	0	1	2	3	4	5	6	…	10	…	100	…
B_n	1	2	5	10	17							

c.

n	0	1	2	3	4	5	6	…	10	…	100	…
C_n	3	7	11	15	19							

Investigation 2 Some Sums

Sometimes it is useful to sum the numbers in a sequence, which is the focus of this investigation. As you work on the problems of this investigation, look for answers to the following questions:

What are strategies for summing the terms of arithmetic and geometric sequences?

What are formulas for these sums?

Arithmetic Sums Consider the sky diving situation once again. The table showing the distance fallen during each second is reproduced below.

Sky Diving Free Fall

Time n (in seconds)	1	2	3	4	5	6	7	8	9
Distance Fallen during Second n, D_n (in feet)	16	48	80	112	144	176	208	240	272

1 From the previous investigation, you know that a function formula for this sequence is

$$D_n = 16 + 32(n - 1).$$

This formula gives the distance fallen *during the nth second*. Use the function formula for D_n to:

a. verify the entries in the table for D_1 and D_5.

b. compute D_{18}, D_{19}, and D_{20}.

2 A sky diver would be very interested in knowing the *total distance* fallen *after n seconds*.

a. What is the total distance fallen after 3 seconds? After 5 seconds?

b. How could you determine the total distance fallen after 20 seconds? Do not actually find this total distance yet. Just explain how you could find it.

c. One approach to finding the sum of the first 20 terms of the sequence D_n is based on a method reportedly discovered by Carl Friedrich Gauss (1777–1855) when he was only 10 years old. Gauss, considered to be one of the greatest mathematicians of all time, noticed that the sum of the terms of an arithmetic sequence, such as $16 + 48 + 80 + \cdots + 560 + 592 + 624$, could be quickly calculated by writing the sum again in reverse order and then adding pairs of corresponding terms, as follows.

$$\begin{array}{r} 16 + 48 + 80 + \cdots + 560 + 592 + 624 \\ \underline{624 + 592 + 560 + \cdots + 80 + 48 + 16} \\ 640 + \phantom{592 + 560 + \cdots + 80 + 48 + 16} \end{array}$$

i. What is the sum of each pair of terms? How many pairs are there?

ii. What is the total distance fallen after 20 seconds?

d. Use Gauss' method, and the function formula for D_n given in Problem 1, to determine the total distance the sky diver would fall in 30 seconds.

e. An expert sky diver typically free-falls to about 2,000 feet above Earth's surface before pulling the rip cord on the parachute. If the altitude of the airplane was about 5,000 feet when the sky diver began the jump, how much time can the sky diver safely allow for the free-fall portion of the flight?

3 You can use Gauss' idea in Problem 2 and algebraic reasoning to derive a general formula for the sum of the terms of an arithmetic sequence with common difference d.

a. If $a_1, a_2, \dots, a_n$ is an arithmetic sequence with common difference d, explain why the sum S_n of the terms a_1 through a_n can be expressed by the formula

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n.$$

b. If you rewrite S_n in reverse order and then add pairs of corresponding terms as in Problem 2, what is the sum of each pair of terms? How many pairs are there?

c. Use your answers in Part b to write a formula for S_n in terms of a_1, a_n, and n. Compare your formula to those of others. Resolve any differences.

d. Sondra developed the following formula-in-words for the sum of the terms of an arithmetic sequence.

$$S = \frac{(\textit{initial term} + \textit{final term})(\textit{number of terms})}{2}$$

Explain why this formula makes sense.

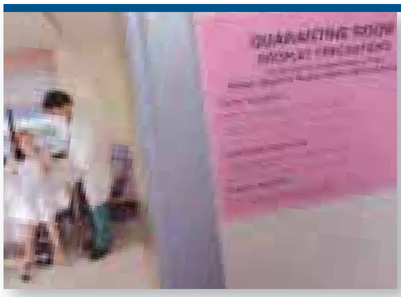
4 As you discovered in the last investigation, the starting subscript used in a sequence can affect the formulas for the sequence. An important feature of the formula-in-words in Part d of Problem 3 is that it does not depend on the subscript notation used for the sequence. In contrast, the formula in Part c of Problem 3 only works if the starting subscript is 1. Consider what happens when the starting subscript is 0.

a. Suppose $a_0, a_1, \dots, a_n$ is an arithmetic sequence. Find a formula for the sum S_n of the terms a_0 through a_n. Your formula should use a_0, a_n and n.

b. Find the sum of the terms of this sequence: 7, 12, 17, 22, … , 52.

Accumulated versus Additional Amount In modeling a situation involving sequential change, it is important to decide whether the situation involves a pattern of change in *additional* amount or in *accumulated* amount.

5 It may happen that in two different situations, you get the same sequence of numbers, but it makes sense to add the terms of the sequence in only one of the situations. Consider an epidemic that begins at Day 1 with one infected person and spreads rapidly through two counties.



a. In Adair County, a health official states, "The population of sick people triples every day." The table below can be used to represent this situation.

Day	1	2	3	4	5	6	7	8	9
Number of Sick People	1	3	9						

What is the total number of sick people at the end of Day 6?

b. In Benton County, a different health official states, "The number of new sick people triples every day." Given this statement, you could construct the same table as in Part a. In this case, what is the total number of sick people at the end of Day 6 (assuming no sick person gets well in that time)?

c. Look back at Parts a and b. In which situation do the terms in the sequence represent an *additional* amount? In which situation do the terms represent an *accumulated* amount? Explain why it makes sense to sum the terms of the sequence in one situation but not in the other.

Geometric Sums In the epidemic example from Problem 5, assume that the epidemic begins at Day 1 with one sick person and the *additional* number of sick people triples every day. Then the total number of sick people at the end of Day n is found by summing the terms in the sequence.

6 Using algebraic reasoning, you can derive a formula for quickly calculating this geometric sum, as outlined below.

a. The total number of sick people at the end of day n can be represented as

$$S_n = 1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-1}.$$

Explain why the exponent on the final term is $n - 1$ and not n.

b. The formula in Part a can be used to compute the sum, but it would take many steps for a large value of n. Derive a shorter formula by carrying out the following plan.

Create a new formula by multiplying both sides of the formula in Part a by 3. Subtract the formula in Part a from your new formula. Then solve the resulting formula for S_n.

c. Compare your work in Part b to the derivation below. Provide reasons for each step in the derivation below.

$$\begin{aligned} S_n &= 1 + 3 + 3^2 + 3^3 + 3^4 + \cdots + 3^{n-1} && (1)\\ 3(S_n) &= 3(1 + 3 + 3^2 + 3^3 + 3^4 + \cdots + 3^{n-1}) && (2)\\ &= 3 + 3^2 + 3^3 + 3^4 + \cdots + 3^{n-1} + 3^n && (3)\\ 3S_n - S_n &= 3^n - 1 && (4)\\ (3-1)S_n &= 3^n - 1 && (5)\\ S_n &= \frac{3^n - 1}{3 - 1} && (6) \end{aligned}$$

d. Explain why the strategy in Part c works to create a short formula.

e. Use this formula to calculate the total number of sick people at the end of Day 6. Compare your answer to your response to Problem 5 Part b.

f. Using similar reasoning, prove that the *sum of the terms of the geometric sequence* $1, r, r^2, r^3, r^4, \ldots, r^{n-1}$, where $r \neq 1$, is

$$1 + r + r^2 + r^3 + r^4 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}.$$

7 When summing terms in a geometric sequence, the exponent of the last term could be any positive integer. In Part f of Problem 6, the exponent of r in the last term is $n - 1$; and then the exponent of r in the numerator on the right side of the sum formula is n. Consider other possibilities.

a. Complete the formula for this sum.
$1 + r + r^2 + r^3 + r^4 + \cdots + r^n = \ldots$

b. Complete the formula for this sum.
$1 + r + r^2 + r^3 + r^4 + \cdots + r^{n+1} = \ldots$

c. It can get very confusing keeping track of when the formula should use n, $n - 1$, $n + 1$, and so on. But the pattern is always the same. State in words the relationship between the ending exponent on r in the sum in the left side of the formulas above and the exponent on r in the numerator of the right side.

8 The sums and formulas you have worked with so far apply to sequences with initial term 1. Consider sums of geometric sequences when the initial term is not 1.

a. Suppose an epidemic begins with two infected people on Day 1 and the additional number of infected people triples every day. What is the total number of sick people at the end of Day 4?

b. Suppose an ice cream store sold 22,000 ice cream cones in 2008. Based on sales data from other stores in similar locations, the manager predicts that the number of ice cream cones sold each year will increase by 5% each year. Using the formulas for the sum of a geometric sequence that you developed in Problems 6 and 7, find the total predicted number of ice cream cones sold during the period from 2008 to 2014. (*Hint*: You may find it helpful to write the sum term-by-term and then factor out 22,000.)

c. Consider a general geometric sequence with common ratio r, where $r \neq 1$, and initial term b. Find a formula for the sum, $S_n = b + br + br^2 + \cdots + br^n$. Provide an argument for why your formula is correct.

9 An expression showing the terms of a sequence added together is called a **series**. For example, $a_0 + a_1 + a_2 + \cdots + a_n$ and $B_1 + B_2 + \cdots + B_n$ are both examples of a series. A series is called an **arithmetic series** or a **geometric series**, depending on whether the sequence that generates the terms is arithmetic or geometric, respectively. (Note that the terms "sequence" and "series" have very specific meanings in mathematics, which may be different than their English meanings.)

i. Write a formula for an arithmetic series $B_1 + B_2 + \cdots + B_n$.

ii. Write a formula for a geometric series $B_1 + B_2 + \cdots + B_n$, with common ratio r.

Summarize the Mathematics

In this investigation, you examined total change in situations involving arithmetic growth and geometric growth or decay.

a Consider the following types of sequential growth models, in which the goal is to find the total amount after a certain number of steps.

Model 1 The growth sequence is arithmetic, and each term in the sequence represents an accumulated amount

Model 2 The growth sequence is arithmetic, and each term in the sequence represents an additional amount

Model 3 The growth sequence is geometric, and each term in the sequence represents an accumulated amount

Model 4 The growth sequence is geometric, and each term in the sequence represents an additional amount

i. Give an example of each of the four sequential growth situations.

ii. In which situations would you sum the sequence to find the total amount after a certain number of steps?

b Suppose an arithmetic sequence has initial term a_0 and common difference d.

i. Write recursive and function formulas for the sequence.

ii. Write a formula for the sum of the terms up through a_n.

c Suppose a geometric sequence has initial term a_0 and common ratio r.

i. Write recursive and function formulas for the sequence.

ii. Write a formula for the sum of the terms up through a_n.

Be prepared to explain your thinking and formulas to the entire class.

Check Your Understanding

Find the indicated sum for each of the sequences below.

a. If $t_1 = 8$ and $d = 3$ in an arithmetic sequence, find the sum of the terms up through t_{10}.

b. If $t_0 = 1$ and $r = 2.5$ in a geometric sequence, find the sum of the terms up through t_{12}.

c. Find the sum of the first 15 terms of each sequence below.

n	1	2	3	4	5	6	7	…	15	…
A_n	2	6	18	54	162					

n	0	1	2	3	4	5	6	…	14	…
B_n	95	90	85	80	75					

d. A popular shoe store sold 5,700 pairs of athletic shoes in 2008. Projections are for a 3% increase in sales each year for several years after 2008. What is the projected total sales (number of athletic shoes sold) during the period from 2008 to 2013?

Investigation 3 Finite Differences

So far, you have been able to find a function formula for a sequence by detecting and generalizing a pattern. Sometimes that is not so easily done. In this investigation, you will learn how to use *finite differences tables* to find a function formula for certain sequences. As you work on the problems of this investigation, look for answers to these questions:

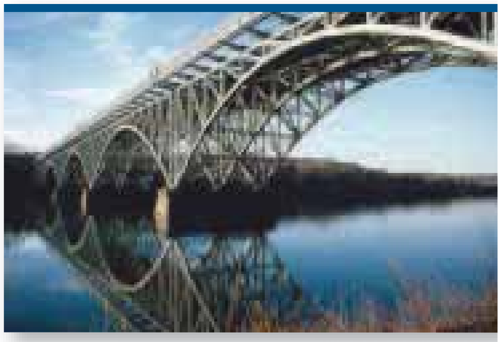
How do you construct a finite differences table for a sequence?

How can you use such a table to find a function formula for a sequence?

What kind of function formulas can be found using finite differences tables?

1 To learn how to use finite differences tables, start with a sample function formula. Suppose you toss a rock straight down from a high bridge. If the rock is thrown from a bridge 300 feet above the river with an initial velocity of 10 feet per second, then the distance D_n of the rock from the river after n seconds is given by the function formula

$$D_n = -16n^2 - 10n + 300.$$



An analysis of the pattern of change in the distance between the rock and the river is shown in the following table.

Number of Seconds, n	D_n	1st Difference	2nd Difference
0	300		
		−26	
1	274		−32
		−58	
2	216		___
		___	
3	___		___
		___	
4	___		

a. On a copy of the table, complete the D_n column.

b. The third column contains differences between consecutive terms of the sequence D_n. How was the "−58" calculated? Find the remaining entries in the "1st Difference" column.

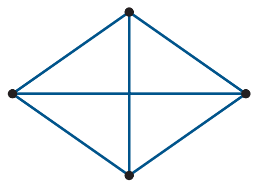
c. The fourth column contains differences between consecutive terms of the "1st Difference" column. Verify the entry "−32" and find the remaining entries in the "2nd Difference" column.

d. A table like this is called a **finite differences table**. Describe any patterns you see in your completed table.

Fact 1 about Finite Differences It is a fact that for any function formula that is a quadratic, such as the one in Problem 1, the 2nd differences in a finite differences table will be a constant.

Conversely, if the 2nd differences are a constant, then the function formula is a quadratic. (See Extensions Task 24.) For example, consider the following counting problem involving vertex-edge graphs.

2 A complete graph is a graph in which there is exactly one edge between every pair of vertices. The diagram at the right shows a complete graph with 4 vertices. In this problem, you will investigate the number of edges E_n in a complete graph with n vertices.



a. On a copy of a table like the one below, enter the number of edges for complete graphs with 0 to 5 vertices. Then compute the 1st and 2nd differences. Describe any patterns you see in the table.

Number of Vertices, n	E_n	1st Difference	2nd Difference
0	___		
		___	
1	___		___
		___	
2	___		___
		___	
3	___		___
		___	
4	___		___
		___	
5	___		

b. What pattern in the finite differences table tells you that the function formula for the sequence of edge counts must look like $E_n = an^2 + bn + c$?

c. How could you find the coefficients a, b, and c? With some classmates, brainstorm some ideas and try them out. When you find a, b, and c, or if you are having trouble finding them, go on to Problem 3.

3 One way to find a, b, and c is to set up and solve a system of three linear equations. If you already did that in Part c of Problem 2, then read through this problem, compare it to your method, and discuss any differences. If you did not set up and solve a system of three linear equations, then do so now, as follows. You can get a system of equations by letting n equal any three values, for example, 1, 2, and 3.

a. To get the first equation, suppose $n = 1$. If you substitute $n = 1$ in the equation $E_n = an^2 + bn + c$, you get $E_1 = a(1)^2 + b(1) + c = a + b + c$. Explain why one equation is $a + b + c = 0$.

b. Use similar reasoning with $n = 2$ and $n = 3$ to get the second and third equations.

c. Compare the system of three linear equations you found to the systems found by others. Resolve any differences.

d. You can solve systems of three equations like these using matrices, as you did for the case of systems of two linear equations in Course 2 Unit 2, *Matrix Methods*. Written in matrix form, the system looks like the partially completed matrix equation below. Fill in the missing entries and solve this matrix equation.

$$\begin{bmatrix} 1 & 1 & \underline{\quad} \\ \underline{\quad} & 2 & 1 \\ 9 & \underline{\quad} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \underline{\quad} \end{bmatrix}$$

$$A \cdot X = C$$

e. Use the results of Part d to write the function formula for the number of edges in a complete graph with n vertices. Use this formula to check the entries in the table from Problem 2.

f. Now that you have a function formula, find a recursive formula for the sequence of edge counts. Do so by examining the table in Problem 2.

4 So far in this investigation, you have examined situations in which the function formula is a quadratic. Now consider the case of a higher-degree function formula. Suppose $A_n = 4n^3 + 2n^2 - 5n - 8$. Make a prediction about which column in the finite differences table for this sequence will be constant. Construct the finite differences table to check your conjecture.

Fact 2 about Finite Differences In general, it is possible to compute more than just 2nd or 3rd differences. If the function formula is an rth degree polynomial, then the rth differences will be constant. The converse is also true. These facts can be used to help find function formulas for certain sequences.

Summarize the Mathematics

In this investigation, you learned about finite differences tables.

a. Describe how to construct a finite differences table for a sequence of numbers.

b. If the 4th differences in the finite differences table for a sequence are constant, what do you think the function formula for the sequence will look like? How would you go about finding the function formula?

c. In general, what kind of function formulas can be found using finite differences tables?

d. The title of this lesson is "A Recursive View of Functions." Explain how this title describes the mathematics you have been studying in all three investigations of the lesson.

Be prepared to share your descriptions and thinking with the entire class.

Check Your Understanding

Use a finite differences table and matrices to find a function formula for the sequence below.

n	0	1	2	3	4	5	6	7	8	9	10	11
A_n	3	12	25	42	63	88	117	150	187	228	273	322

On Your Own

Applications

1 For many people, a college education is a desirable and worthwhile goal. But the cost of a college education is growing every year. For the 2006–07 school year, the average cost for four-year public colleges (tuition and fees) was \$5,836, which was up 6.3% from the previous year. (Source: College Board; www.collegeboard.com/student/pay/add-it-up/4494.html)

a. Assume an annual increase rate in college costs of 6% per year. Make a table showing the average cost of a year of college education (tuition and fees) at a four-year public college for the next 5 years following 2006–07.

b. Is the sequence of increasing costs arithmetic, geometric, or neither?

c. Determine recursive and function formulas for the sequence of costs.

d. Use your formulas to predict the average cost of the first year of your college education if you go to a four-year public college right after you graduate from high school.

e. Predict the average cost of four years of college at a four-year public college for a child born this year.

2 Animal behavior often changes as the outside temperature changes. One curious example of this is the fact that the frequency of cricket chirps varies with the outside temperature in a very predictable way. Consider the data below for one species of cricket.

Cricket Chirps

Temperature (in °F)	45	47	50	52	54	55	60
Frequency (in chirps/min)	20	28	41	50	57	61	80

a. If you were to choose an arithmetic sequence or a geometric sequence as a model for the frequency sequence, which type of sequence would you choose? Why?

b. As is often the case in mathematical modeling, the model you chose in Part a does not fit the data exactly. Nevertheless, it may be quite useful for analysis of the situation.

  i. Find a recursive formula for the frequency sequence, based on the type of sequence you chose in Part a.

  ii. Use the formula to predict the frequency of cricket chirps for a temperature of 75°F.

c. At what temperature would you expect crickets to stop chirping?

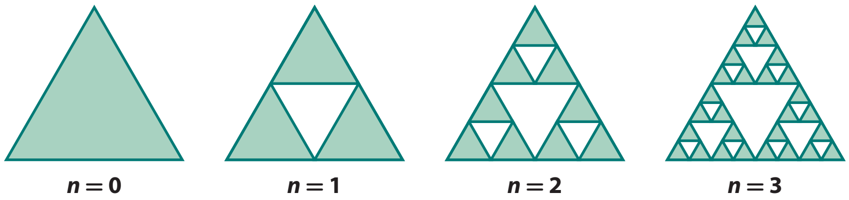
d. You can use the relationship between the frequency of cricket chirps and the temperature as a kind of thermometer. What would you estimate the temperature to be if a cricket is chirping at 100 chirps per minute?

e. Based on your recursive formula in Part b, find a function formula for the sequence of cricket chirps. Use your function formula to find the frequency of cricket chirps for a temperature of 75°F. Compare to your answer in Part b.

f. Rewrite your function formula to express temperature as a function of frequency of cricket chirps. Use this new formula to answer the question in Part d.

3 The square Sierpinski carpet you examined in Investigation 1 is an example of a *fractal* in that small pieces of the design are similar to the design as a whole. Other fractal shapes can also be constructed using recursive procedures.

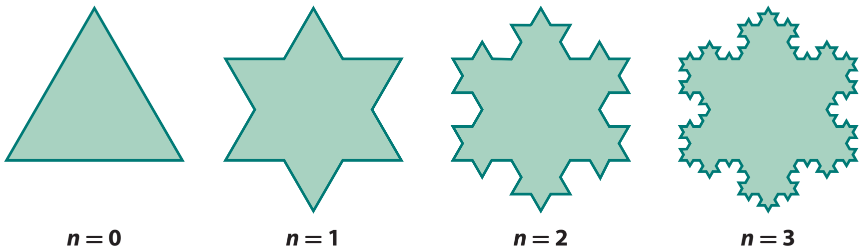
a. A *Sierpinski triangle* is constructed through a sequence of steps illustrated by the figures below.



At stage $n = 0$, you construct an equilateral triangle whose sides are all of length 1 unit. In succeeding stages, you remove the "middle triangles," as shown in stages $n = 1$, 2, and 3. This process continues indefinitely. Consider the sequence of areas of the figures at each stage. Find recursive and function formulas for the sequence of areas.

b. Another interesting fractal is the *Koch snowflake*. The procedure for constructing the Koch snowflake also begins with an equilateral triangle whose sides are of length 1 unit.

At each stage, you remove the segment that is the middle third of each side and replace it with two outward-extending segments of the same length, creating new equilateral triangles on each side as shown in the diagrams below. The process continues indefinitely.



Find recursive and function formulas for the sequence of perimeters of the snowflake.

4 *Glottochronology* is the study of changes in languages. Over time, certain words in a language are no longer used. In effect, they disappear from the language. Suppose a linguist examines a list of 500 words used in a language 1,000 years ago. Let $W(n)$ be the percentage of the words in this list that are still in use n years later, given as a decimal.

It is commonly assumed that $W(n)$ is proportional to $W(n - 1)$. Glottochronologists have determined that the constant of proportionality can be estimated to be about 0.99978. (Problem adapted from Sandefur, James T. *Discrete Dynamical Modeling*. Oxford, 1993, page 63.)

a. Find recursive and function formulas for this sequence.

b. According to this model, about how many of the 500 words are still in use today?

5 Irene has a sales job which pays her a monthly commission. She made $250 in her first month. Her supervisor tells her that she should be able to increase her commission income by 10% each month for the next year. For this task, assume that the supervisor's prediction is correct.

a. Find recursive and function formulas for the sequence of monthly commission income.

b. How much total commission income will Irene earn in her first ten months on the job?

6 As you complete this task, think about the defining characteristics of arithmetic and geometric sequences and how those characteristics are related to the sum of their terms. For each sequence below, determine if the sequence is arithmetic, geometric, or neither. Then find the sum of the indicated terms.

a. 13, 26, 52, 104, … , 6,656

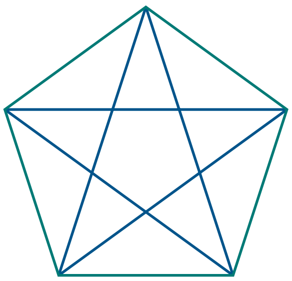
b. 13, 12.25, 11.5, 10.75, … , 1

c. 1, 4, 9, 16, … , 225

d. 5, 4, 3.2, 2.56, … , 0.8388608

7 In this task, you will use the idea of sequential change to investigate the number of diagonals that can be drawn in a regular n-sided polygon.

a. Draw the first few regular n-gons and make a table showing the number of diagonals that can be drawn in each of them.



b. Determine a recursive formula for the number of diagonals that can be drawn in a regular n-gon.

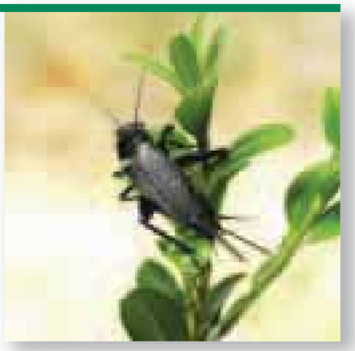
c. Use a finite differences table to find a function formula for the number of diagonals that can be drawn in a regular n-gon.

d. Find the number of diagonals in a regular 20-gon.

e. What other methods might you use to find a function formula for the sequence in Part a?

8 Use a finite differences table and matrices to find the function formula for the sequence given by $B_n = B_{n-1} + 3n$, with $B_0 = 2$.

Connections



9 In Applications Task 2, you may have established that the sequence of chirping frequencies for one species of cricket is approximately an arithmetic sequence with recursive formula

$$C_n = C_{n-1} + 4.$$

a. Plot the (*temperature, frequency*) data from the table on page 499. Find a regression equation that fits the data.

b. Using the equation from Part a, write a function formula for the sequence. Check that your formula generates the values given in the table for the chirping frequencies at temperatures of 47°F and 60°F.

10 In Course 2 Unit 8, *Probability Distributions*, you investigated the waiting-time distribution in the context of a modified Monopoly® game in which 36 students are in jail and a student must roll doubles to get out of jail.

a. The probability of rolling doubles is $\frac{1}{6}$. Thus, the probability of getting out of jail on any given roll of the dice is $\frac{1}{6}$. What is the probability of remaining in jail on any given roll of the dice?

b. Complete a copy of the following table. The waiting-time distribution refers to the first two columns, since you want to know how long a student has to wait to get out of jail. The last column provides important related information about the number of students still in jail.

Rolling Dice to Get Doubles

Number of Rolls to Get Doubles	Expected Number of Students Released on the Given Number of Rolls	Expected Number of Students Still in Jail
1	6	30
2	5	
3		
4		
5		

c. Consider the sequence of numbers in the last column.

i. Is the sequence an arithmetic or geometric sequence? Why?

ii. Determine the recursive and function formulas for the sequence. (Use 36 as the initial term of the sequence since that is the initial number of students in jail.)

iii. Use the formulas to find the expected number of students left in jail after 20 rolls.

d. Sketch a histogram for the waiting-time distribution shown in the first two columns.

i. Write a recursive formula that shows how the height of any given bar compares to the height of the previous bar.

ii. What kind of sequence is the sequence of bar heights?

e. A waiting-time distribution is also called a *geometric distribution*. Explain why the use of the word "geometric" for this type of distribution seems appropriate in terms of sequences.

11 In this lesson, you found function formulas for arithmetic and geometric sequences. In Lesson 1, you studied combined recursive formulas of the form $A_n = rA_{n-1} + b$. You can now use what you know about geometric sequences and their sums to find a function formula for such combined recursive formulas.

a. Complete the following list of terms for the combined recursive formula $A_n = rA_{n-1} + b$. Examine the list for patterns so that you can write a function formula for A_n.

$A_0 = A_0$
$A_1 = rA_0 + b$
$A_2 = rA_1 + b = r(rA_0 + b) + b = r^2A_0 + rb + b$
$A_3 = r(r^2A_0 + rb + b) + b = r^3A_0 + r^2b + rb + b$
$A_4 = ?$
$A_5 = ?$
$\vdots$
$A_n = ?$

b. The expression you have for A_n at the end of Part a is a function formula, but it can be simplified. To simplify, use what you know about the sum of the terms of a geometric sequence.

c. Now reconsider the combined recursive formula that modeled the fish-population problem in Lesson 1: $A_n = 0.8A_{n-1} + 1{,}000$, with $A_0 = 3{,}000$.

i. Use the general function formula from Part b to write a function formula that models the fish population.

ii. Now use this formula to find A_2 and A_{10}. Choose a year far in the future, and find the long-term population. Compare your results to those you found in Lesson 1, using other methods.

d. Suppose the combined recursive formula $A_n = rA_{n-1} + b$ represents year-to-year population change. Assume that r is a positive number less than 1. Use the function formula from Part b to explain why the long-term population in this situation is $\frac{b}{1-r}$. Use this fact to find the long-term population in the fish-population problem, and compare it to what you had previously found.

e. Compare the function formula you obtained in Part c to the one given in Connections Task 7 of Lesson 1, page 472. Explain and resolve any differences.

12 You have seen that a sequence can be represented with function notation. There is an even stronger connection between sequences and functions. In fact, a finite sequence can be defined as a function whose domain is the set $\{1, 2, \ldots, n\}$ and whose range is a set containing the terms of the sequence. For example, the sequence $7, 10, 13, \ldots, 7 + 3(n - 1)$ can be defined as a function, s, from the set $\{1, 2, \ldots, n\}$ to the set $\{7, 10, 13, \ldots, 7 + 3(n - 1)\}$, where $s(1) = 7$, $s(2) = 10$, … , and $s(n) = 7 + 3(n - 1)$.

a. Consider the sequence $3, 6, 12, 24, \ldots, 3(2)^{n-1}$. Describe how this sequence can be defined as a function. Describe the domain and range as sets.

b. When using functions to define sequences, the domain can actually be any finite set of integers. In particular, the domain might be $\{0, 1, 2, \ldots, n\}$. Give an example of a sequence that could be defined as a function whose domain is $\{0, 1, 2, \ldots, n\}$.

13 A series $a_0 + a_1 + a_2 + \cdots + a_n$ can be written in a more compact form using sigma notation as $\sum_{i=0}^{n} a_i$. You have previously used sigma notation (without subscripts) when writing a formula for standard deviation and when considering Pearson's correlation as well as other occasions.

a. Suppose $x_1, x_2, x_3, \ldots, x_n$ are sample data values. Use sigma notation with subscripts to write an expression for:

i. the mean of this distribution.

ii. the standard deviation of this distribution.

b. Write $\sum_{k=0}^{12} 3k$ in expanded form and then find the sum.

c. Write $\sum_{i=3}^{10} 2^i$ in expanded form and then find the sum.

d. Complete the formula below by determining the subscripts.

$$3 + 5 + 7 + 9 + 11 + 13 + 15 = \sum_{n=?}^{?} (2n + 1)$$

14 Below is the sequence from the Check Your Understanding task on page 504.

n	0	1	2	3	4	5	6	7	8	9	10	11
A_n	3	12	25	42	63	88	117	150	187	228	273	322

a. Produce a scatterplot of n versus A_n using a graphing calculator or computer software. Describe the shape of the graph. Which function family would best model this data pattern?

b. Find a regression equation that seems to best fit the scatterplot. Compare this equation to the function formula you derived in the Check Your Understanding task using a finite differences table. Describe and resolve any differences in the two solutions.

c. Do you think statistical regression methods will work to find a function formula for any sequence? Explain.

15 In Investigation 2, you considered two sequences related to the free fall of a sky diver: the sequence D_n of distance fallen *during* each second and the sequence T_n of total distance fallen *after* n seconds. Continue analyzing these sequences.

a. Use the information given in the Think About this Situation on page 482 to verify that the two sequences below, D_n and T_n, are accurate models if you assume Earth's gravity and no air resistance.

Distance Fallen (in feet)

n	D_n	T_n
1	16	16
2	48	64
3	80	144
4	112	256
5	144	400
6	176	576
7	208	784
8	240	1,024
9	272	1,296
10	304	1,600

b. Use a finite differences table to find a function formula that describes the sequence of total distance fallen.

c. Use calculator- or computer-based regression methods to find a function formula relating total distance fallen to the number of seconds fallen.

d. For the Think About This Situation questions, you wrote recursive and function formulas for the sequence D_n. Note that each term (after the first) of T_n is the result of adding the corresponding term of D_n to the previous term of T_n. Use this fact to help you find a recursive formula for T_n.

Reflections

16 Suppose you have both a recursive formula and a function formula for a sequence of numbers. Which formula would you use in each of the following situations? Why?

a. Suppose you want to find the 100th term of the sequence of numbers.

b. Suppose you want to find all of the first 100 terms of the sequence.

17 Describe one situation in your daily life or in the daily newspaper that could be modeled by a geometric sequence. Describe another situation that could be modeled by an arithmetic sequence.

18 The English economist Thomas Malthus (1766–1834) is best remembered for his assertion that *food supply grows arithmetically while population grows geometrically*. Do some research, and write a brief paper about this idea. Your essay should address the following questions.

- What is the meaning of Malthus' statement in terms of sequences you have studied?
- Do you think it is a reasonable statement?
- What are its consequences?
- Why has this statement been called "apocalyptic"?
- Have the events of the last 200 years borne out the statement?
- What can we learn from Malthus' statement even if it is not completely accurate?

19 In Problem 3 on page 497, you solved the matrix equation $AX = C$ to find the coefficients of a quadratic equation.

You found that $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}$.

Do you think this matrix will change if you work a similar problem that involves different data? Explain.

Extensions

20 In this unit, you have investigated recursive formulas and sequences that can be modeled by *NOW-NEXT* formulas. This means that to find *NEXT*, you need to use only *NOW*, one step before *NEXT*. However, there are sequences for which finding *NEXT* requires using more than one step before *NEXT*. One of the most famous sequences of this type is called the *Fibonacci sequence*, named after the mathematician who first studied the sequence, Leonardo Fibonacci (c. 1175–c. 1250).

a. Here is the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … . Using the words *NEXT*, *NOW*, and *PREVIOUS*, describe the pattern of the sequence.

b. Let F_{n+1} represent *NEXT*. Use F_{n+1}, F_n, and F_{n-1} to write a recursive formula for the Fibonacci sequence.

c. A recursive formula for a sequence similar to the Fibonacci sequence is $A_n = 2A_{n-1} - A_{n-2}$.

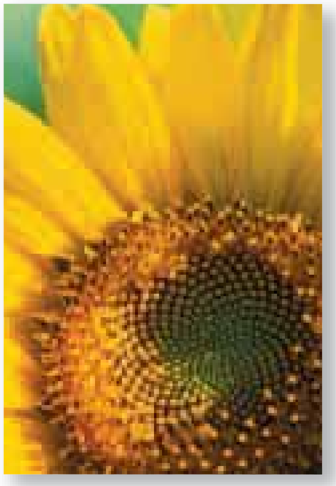
   i. Why can't you list the terms of this sequence?

   ii. Choose two initial values, and list the first six terms of the sequence. Then choose different initial values, and list the first six terms of the sequence. Compare the two sequences. Describe any patterns you see.

   iii. Write a recursive formula for this sequence that uses only A_n, A_{n-1}, A_1, and A_0.

d. Consider the sequence given by $A_n = A_{n-1} + A_{n-2} + 7A_{n-3}$. Choose some initial values. List the first six terms of this sequence.

e. The Fibonacci sequence has many interesting patterns and shows up in the most amazing places. In the photo at the right, the flower has 34 spirals in the counterclockwise direction and 21 in the clockwise direction. These two numbers are successive terms in the Fibonacci sequence. Sunflowers and pine cones also have spirals with numbers from the Fibonacci sequence, as do many other things in nature.



In fact, entire books and journals have been written about this sequence. Find an article or book about the Fibonacci sequence. Write a brief report on one of its patterns or applications.

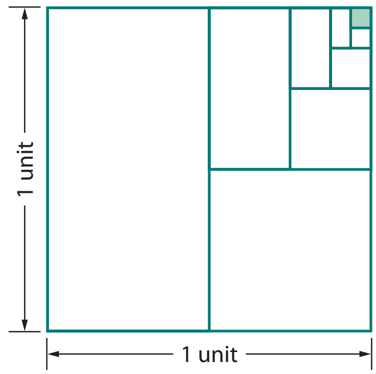
21 In Investigation 2, you found formulas for the sum of a finite number of terms of a geometric sequence. A sequence can also have an infinite number of terms. It is possible to mathematically analyze the sum of an infinite number of terms. Consider this sequence.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

a. Explain why this is a geometric sequence. Find recursive and function formulas for this sequence.

b. Find a formula for the sum of the terms up through $\frac{1}{2^n}$.

c. This sequence has infinitely many terms. Does the geometric model of this sequence shown below suggest what the infinite sum of all the terms of the sequence might be?



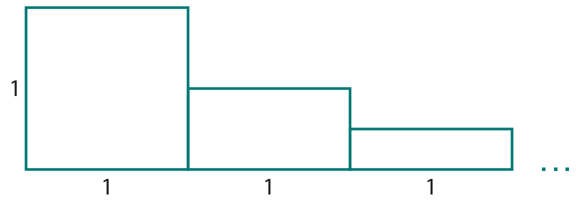
d. A mathematical analysis of infinite sums involves thinking about what happens to the sum of n terms as n gets very large. Examine the formula for the sum of terms up through $\frac{1}{2^n}$ from Part b. What happens to this formula as n gets very large? Your answer should give you the infinite sum.

e. Consider a general geometric sequence with terms t_n and common multiplier r. Suppose that $0 < r < 1$. Determine a general formula for the infinite sum by considering what happens to the finite sum formula if n is very large.

f. Use the general infinite sum formula for $0 < r < 1$ from Part e to find the infinite sum of the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$. Compare the sum to your answer for Part d.

g. Construct a geometric sequence of your choice with $0 < r < 1$. Use the general infinite sum formula from Part e to find the infinite sum.

22 Extensions Task 21 involved infinite geometric sums. Infinite sums lead to some surprising results. Consider a figure that consists of the rectangles of width 1 and height $\frac{1}{2^n}$ arranged as shown below.



There are infinitely many rectangles that comprise this figure. Do you think it is possible for a figure to have finite area but infinite perimeter? Explain using this particular figure.

23 Oranges are one of the most popular foods during Chinese New Year. During the New Year celebration, you may find them in many homes stacked on a platter in the shape of a pyramid with a square base. The number of oranges in such a stack depends on the number of layers in the stack.

a. Complete a table like the one below.

Pyramid of Oranges

Number of Layers in a Stack	0	1	2	3	4	5	6
Number of Oranges in a Stack	0	1	5				

b. Find a recursive formula for the sequence of number of oranges in a stack.

c. Use a finite differences table to find a function formula for the sequence of number of oranges in a stack. How many oranges are in a stack with 15 layers?

24 In Investigation 3, you used a finite differences table to find a function formula for certain sequences. You used the fact that the function formula for a sequence is a quadratic if and only if the 2nd differences in a finite differences table are constant. In this task, you will prove that fact.

a. Suppose a sequence has a quadratic function formula: $A_n = an^2 + bn + c$. Construct a finite differences table. Verify that the 2nd differences are constant. Describe any relationship you see between the constant 2nd differences and the function formula.

b. Now, conversely, suppose the 2nd differences are constant. Provide an argument for why the function formula is a quadratic.

25 Recall the bowhead whale situation from Applications Task 4 in Lesson 1. A status report on the bowhead whales of Alaska estimated that the 1993 population of these whales was between 6,900 and 9,200 and that the difference between births and deaths yielded an annual growth rate of about 3.1%. No hunting of bowhead whales is allowed, except that Alaskan Inuit are allowed to take, or harvest, about 50 bowhead whales each year for their livelihood. (Source: nmml.afsc.noaa.gov/CetaceanAssessment/ bowhead/bmsos.htm)

a. Use 1993 as the initial year, Year 0. The initial population is a range of values, rather than a single value. Specifically, the range of initial population values is 6,900 to 9,200. Let $L(n)$ be the lower population value in the range of values in year n. Similarly, let $H(n)$ be the higher population value in the range of values in year n. Find recursive formulas for $L(n)$ and $H(n)$.

b. Find a recursive formula and a function formula for $S(n) = H(n) - L(n)$. Describe how the size of the population range changes over time.

c. The recursive formulas for $H(n)$ and $L(n)$ that you found in Part a are combined recursive formulas of the form $A_n = rA_{n-1} + b$. Thus, as in Connections Task 11 (page 503), it is possible to find a function formula for $H(n)$ and $L(n)$ by listing the terms and summing a geometric series. This yields the following formulas.

$$L(n) = 1.031^n(6{,}900) - 50\left(\frac{1.031^n - 1}{1.031 - 1}\right)$$

$$H(n) = 1.031^n(9{,}200) - 50\left(\frac{1.031^n - 1}{1.031 - 1}\right)$$

Verify that the above two formulas can be transformed to the two equivalent formulas below.

$$L(n) = 1.031^n\left(6{,}900 - \frac{50}{0.031}\right) + \left(\frac{50}{0.031}\right)$$

$$H(n) = 1.031^n\left(9{,}200 - \frac{50}{0.031}\right) + \left(\frac{50}{0.031}\right)$$

d. In Part b, you considered the difference between the higher and lower values of the population range. Now consider the ratio of the higher value to the lower value.

i. Use the recursive formulas in Part a, along with a spreadsheet or other technology tool, to create a table of values for the ratio of higher value to lower value. Describe any patterns that you see in the table, particularly the long-term trend.

ii. Use the formulas in Part c to construct a function formula for the ratio of higher values to lower values. Estimate the formula if n is very, very large. Use the formula to predict the long-term trend in the ratios. Compare it to the trend that you observed above in part i. Explain any similarities and differences.

Review

26 Rewrite each quadratic expression in vertex form.

a. $x^2 + 8x + 10$

b. $x^2 - 18x - 6$

c. $x^2 + 3x - 9$

27 In 2005, approximately 31% of people 65 or older in the United States used the Internet. Suppose that in 2005, Nate randomly selected 150 people in the United States who were 65 or older.

a. How many of these people do you expect to say that they used the Internet?

b. Is the binomial distribution approximately normal?

c. What is the standard deviation of this distribution? What does it tell you?

d. Would you be surprised to find that 60 of the people in Nate's survey said that they used the Internet? Explain your reasoning.

e. Use a standardized value to estimate the probability that only 40 of the people in Nate's study used the Internet.

28 Consider the statement: If quadrilateral *ABCD* is a parallelogram, then $\overline{BD}$ divides the quadrilateral into two congruent triangles.

a. Is this a true statement? If yes, prove the statement. If not, provide a counterexample.

b. Write the converse of this statement.

c. Is the converse true? If yes, provide a proof. If not, provide a counterexample.

29 Write each product or quotient of rational expressions in equivalent form as a single algebraic fraction. Then simplify the result as much as possible.

a. $\dfrac{5x}{x + 4} \cdot \dfrac{3x + 12}{x^3}$

b. $\dfrac{x - 4}{2} \cdot \dfrac{6}{4 - x}$

c. $\dfrac{x^2 + x}{5} \div \dfrac{2x + 2}{15}$

d. $\dfrac{12x - 6}{x + 1} \div \dfrac{6x - 3}{x}$

30 Find the solution set for each inequality. Represent the solution set using a number line, symbols, and interval notation.

a. $x^2 - 5x > x + 7$

b. $\frac{4}{x} > 3 - x$

31 The spinner below is used at the community fair in Lennox each year. The spinner is divided into eight equal sections. Suppose Sonia buys two spins of the spinner.

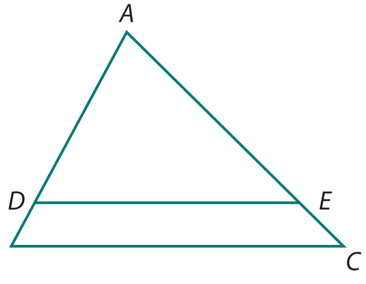
a. What is the probability that she will not win a prize?

b. What is the probability that she will win two cookies?

c. What is the probability that she will win a cookie and a drink?

d. What is the probability that she will win two drinks?

32 In the triangle below, $\overline{DE} \parallel \overline{BC}$.



a. Explain why $\frac{AB}{AD} = \frac{AC}{AE}$.

b. Suppose that $AD = 10$ cm, $AB = 12$ cm, and $AE = 15$ cm. Find AC and EC.

c. For the lengths in Part b, is it the case that $\frac{DB}{AD} = \frac{EC}{AE}$?

d. Provide reasons for each step in the proof below that Jamal wrote to show that $\frac{DB}{AD} = \frac{EC}{AE}$ is always true given $\overline{DE} \parallel \overline{BC}$. His proof begins with the result from Part a.

$$\frac{AB}{AD} = \frac{AC}{AE}$$

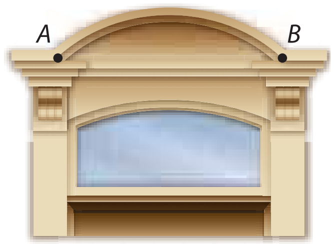
$$\frac{AD + DB}{AD} = \frac{AE + EC}{AE}$$

$$1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$$

$$\frac{DB}{AD} = \frac{EC}{AE}$$

e. Suppose that $AD = 8$ ft, $DB = 2$ ft, and $AE = 18$ ft. How long is $\overline{EC}$?

33 The arch above the portion of a door shown at the right is part of a circular region bounded by a chord and part of a circle. The chord is 40 inches long and the height of the arch is 8 inches. The radius of the circle that forms $\overset{\frown}{AB}$ is 29 inches. What is the measure of $\overset{\frown}{AB}$?



34 Determine the number and type (integer, noninteger rational, irrational, or nonreal complex) of solutions for each quadratic equation.

a. $3x^2 - 5x + 2 = 0$

b. $4x^2 + 25 = 20x$

c. $x^2 + 6x = 10$

35 As part of the preparation for the 2008 Beijing Olympics, a great observation wheel with a diameter of 198 meters was built. The wheel has 48 capsules which are evenly spaced around the outside of the wheel.

a. What is the distance from one capsule to the next along the outside of the wheel?

b. What is the shortest distance between two adjacent capsules?

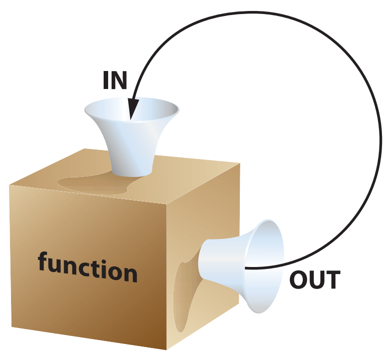
c. If the wheel completes one revolution every 30 minutes, what is the linear velocity of the wheel in kilometers per hour?

LESSON 3

Iterating Functions

Function iteration is a relatively new field of mathematical study, with many unanswered questions and connections to contemporary mathematical topics such as fractal geometry and chaos theory. Applications of function iteration are being discovered every day, in areas like electronic transmission of large blocks of data, computer graphics, and modeling population growth.

A function can be thought of as a machine that accepts inputs and produces outputs. For example, for the function $f(x) = x^2$, an input of 2 produces an output of 4. To begin understanding function iteration, imagine starting with a specific input, such as 2, and then sequentially feeding the outputs back into the function as new inputs.



Think About This Situation

Think about the function $f(x) = x^2$ as an input-output machine.

a Start with an input of 2. The resulting output is 4.

- Put 4 in as a new input. What is the new output?
- Put the new output in as the next input. What is the next output?
- Continue this process of feeding outputs back in as new inputs for several more steps.
- What do you think will be the long-term behavior of the sequence of numbers generated by this process?

b Suppose you carry out the same process for the same function, except this time start with an input of $\frac{1}{2}$. What do you think will be the long-term behavior of the resulting sequence of numbers in this case?

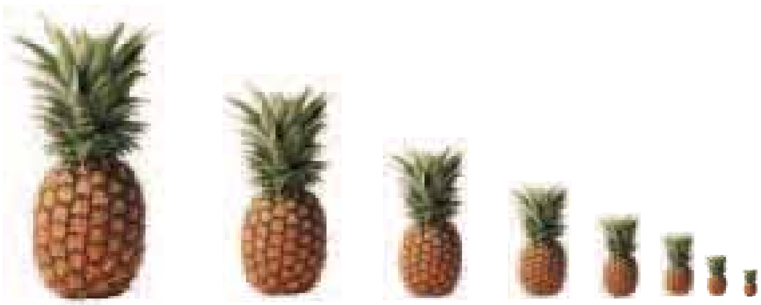
c What happens when you carry out the same process with a starting input of $x = 1$?

d In what way is this process of feeding outputs back in as inputs a recursive process?

Functions and recursive thinking are both important and unifying ideas in *Core-Plus Mathematics*. In this lesson, you will explore a connection between recursive formulas and function iteration. You will use this connection to further analyze processes of sequential change.

Investigation 1 Play It Again ... and Again

The process of feeding the outputs of a function back into itself as inputs is called **iterating a function**. Imagine making a reduced copy of an image, then making a reduced copy of your copy, then making a reduced copy of that copy, and so on.



As you work on the problems of this investigation, look for answers to the following questions:

How do you iterate a function, and how can technology help?

What are connections between recursive formulas and function iteration?

What are some possibilities for long-term behavior in function iteration sequences?

Connection between Recursive Formulas and Function Iteration As you might expect, there is a close connection between iterating a function and evaluating a recursive formula.

1. Consider the rule $NEXT = 2(NOW)^2 - 5$.

   a. Use an initial value of 1 and find the next three values.

   b. Rewrite the rule as a recursive formula using U_n and U_{n-1}. Let $U_0 = 1$ and then find U_1, U_2, and U_3.

   c. Now think about iterating a function, as illustrated in the "function machine" diagram on page 514. Iterate $f(x) = 2x^2 - 5$ three times, starting with $x = 1$.

   d. Compare the sequences of numbers you got in Parts a, b, and c. Explain why the three sequences are the same, even though the representations used to generate the sequences are different—using *NOW-NEXT* in Part a, using U_n and U_{n-1} in Part b, and iterating a function in Part c. (If the three sequences you generated in Parts a, b, and c are not the same, go back and examine your work, compare to other students, and resolve any problems.)

2. Think about iterating the function $f(x) = 3x + 1$. A table that shows the iteration process is similar to function tables that you have previously used but with a new twist.

   a. Complete a table like the one below, starting with $x = 2$.

Iterating $f(x) = 3x + 1$

	x IN	$f(x)$ OUT
Start →	2	7
	7	22
	?	?
	?	?
	?	?
	?	?

   b. How would the table be different if you started with $x = 0$?

   c. What recursive formula yields the same sequence of numbers as that generated by iterating $f(x)$?

3. Consider the recursive formula $U_n = (U_{n-1})^2 + 3U_{n-1} + 4$, with $U_0 = 1$.

   a. Compute U_1, U_2, and U_3.

   b. Rewrite the recursive formula as a rule using the words *NOW* and *NEXT*.

   c. What function can be iterated to yield the same sequence of numbers as that generated by the recursive formula? Check your answer by iterating your function, starting with an input of 1, and comparing to your answers for Part a.

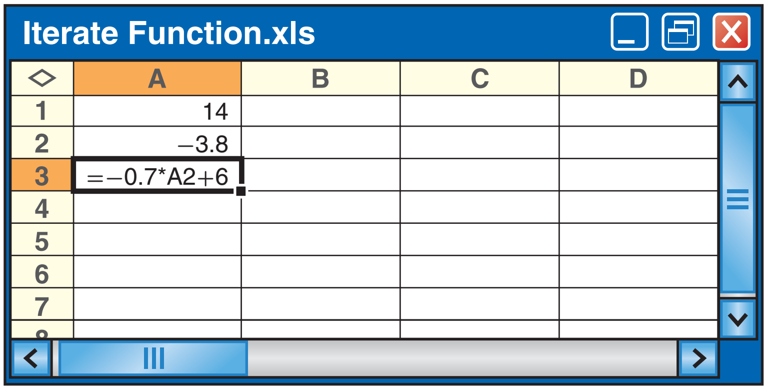
Using Technology to Iterate Functions Use of technology such as calculators and spreadsheets can be very helpful when iterating functions.

CPMP-Tools

4. Consider the function $g(x) = -0.7x + 6$.

   a. Complete a table like the one in Problem 2 showing the first few steps of iterating $g(x)$, starting with $x = 14$.

   b. Use a spreadsheet to iterate $g(x)$ at least 30 times, starting with 14. Describe any patterns you see in the iteration sequence.



   c. Use the last-answer feature of a calculator to iterate $g(x)$ at least 30 times, starting with 14. Make sure you get the same sequence of numbers as in Part b.

   d. In Parts a, b, and c, you used three methods for iterating $g(x)$: completing a table by hand, using a spreadsheet, and using the last-answer feature on a calculator. What are some advantages and disadvantages of each method for iterating a function?

   e. Use the recursion mode on your calculator or use spreadsheet software to produce a graph illustrating the iteration of $g(x)$, starting with $x = 14$. Describe how the numerical patterns that you found in Part b are illustrated in the graph.

Long-Term Behavior As with any process of sequential change, it is important to study the long-term behavior of function iteration.

5 Suppose $f(x) = \sqrt{x}$. Iterate this function, starting with $x = 256$. Describe the long-term behavior of the resulting sequence of numbers.

6 Suppose $h(x) = 1 - x$. Iterate this function. Describe the long-term behavior of the iteration sequence for each of the following starting values.

a. Start with $x = 5$.

b. Start with $x = -28$.

c. Start with $x = \frac{1}{2}$.

In the next problem, you will iterate functions of the form $f(x) = rx(1 - x)$. This is a broadly useful equation in mathematics called the **logistic equation** (or **logistic map**). For different values of r, you get different functions, each of which may have a different behavior when iterated. Iterating these functions has proven to be a very useful method of modeling certain population growth situations. Also, the study of the iterated behavior of these functions has contributed to many important developments in modern mathematics over the last several decades.

7 For each of the function iterations below, use $x = 0.02$ as the starting value. Describe any patterns you see, including the long-term behavior.

a. Iterate $f(x) = 2.7x(1 - x)$.

b. Iterate $g(x) = 3.2x(1 - x)$.

c. Iterate $h(x) = 3.5x(1 - x)$.

d. Iterate $j(x) = 3.83x(1 - x)$.

e. Iterate $k(x) = 4x(1 - x)$.

Summarize the Mathematics

In this investigation, you explored iteration of linear and nonlinear functions.

a Explain the connection between function iteration and recursive formulas.

b Consider evaluating the rule $NEXT = (NOW)^3 + 5$, starting with 2.

i. What recursive formula (using subscript notation), when evaluated, will produce the same sequence?

ii. What function, when iterated, will generate the same sequence?

c Describe some of the possible long-term behaviors that can occur when a function is iterated.

Be prepared to share your descriptions and thinking with the entire class.

Check Your Understanding

Function iteration can be applied to any function.

a. Iterate $g(x) = \cos x$, starting with $x = 10$. (Use radians, not degrees.) Describe the long-term behavior of the iteration sequence.

i. What recursive formula yields the same sequence as iterating $g(x)$?

ii. Rewrite your recursive formula using the words *NOW* and *NEXT*.

b. Choose any function not used in this investigation and several different starting points. Iterate your function using each of your starting points. Describe the long-term behavior of the iteration sequences.

Investigation 2 Iterating Linear Functions

In Investigation 1, you iterated a variety of functions, both linear and nonlinear. The iterative behavior of nonlinear functions is not yet completely understood and is currently a lively area of mathematical research. On the other hand, iteration of linear functions, which is the focus of this investigation, is well understood. As you work on the problems of this investigation, look for answers to the following questions:

How can you graphically iterate a function?

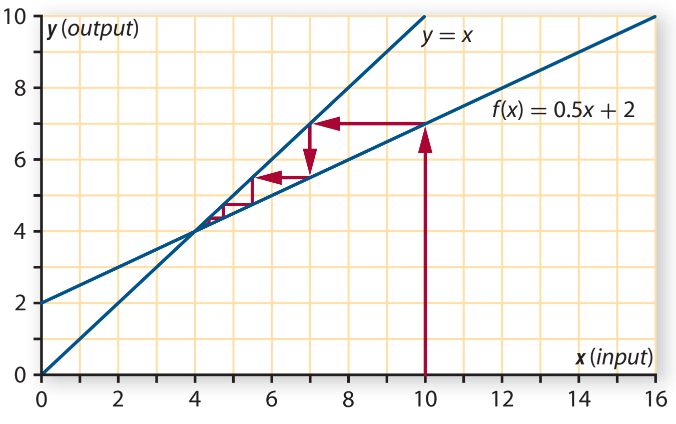
What are all the possible long-term behaviors when iterating linear functions?

How can you use slope to predict these behaviors?

Graphical Iteration Just as with many other ideas in mathematics, the process of function iteration can be represented visually.

To see how graphical iteration works, consider the function $f(x) = 0.5x + 2$. The graphs of $y = x$ and $y = 0.5x + 2$ are shown below on the same set of axes. Think graphically about how an input becomes an output, which then becomes the next input, which produces the next output, and so on.

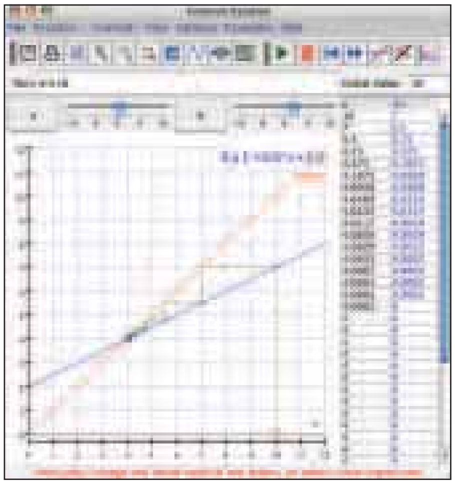
Graphical Function Iteration



In the diagram on the previous page, 10 has been chosen as the original input. To find an input's (x value's) resulting output (y value), you go up to the graph of the function you are iterating. Next, according to the process of function iteration, the output gets put back into the function as an input. This is accomplished graphically by moving horizontally to the $y = x$ graph, since on this line the y value (the current output) is identical to the x value (the new input).

Now go vertically again to find the output associated with the new input. The process continues in this way until, in this example, you are drawn into the intersection point of the two graphs. This is the process of **graphical iteration**.

1 From this graphical perspective, what is the long-term behavior of the function $f(x) = 0.5x + 2$ when iterated? Use your calculator or a spreadsheet to iterate $f(x)$, starting with $x = 10$, and see if the numerical result matches the graphical result.



2 Analyze the process of graphical iteration as illustrated in the diagram on the previous page. If available, use a technology tool such as the "Function Iteration" custom tool in *CPMP-Tools* to illustrate the process.

a. Explain, in your own words, why you can graphically find the output of a function for a given input by moving vertically to the graph of the function.

b. Explain, in your own words, why you can graphically turn an output into the next input by moving horizontally from the graph of the function to the graph of $y = x$.

c. Complete a table like the one in Investigation 1 Problem 2 on page 516 that shows the first few steps of the graphical iteration illustrated above. Explain how the entries in the table correspond to the steps of the graphical iteration process.

d. Illustrate the process of graphical iteration for this function using $x = 1$ as the original input. Describe the pattern of graphical iteration.

3 Sketch graphs and illustrate the process of graphical function iteration for the function $g(x) = -0.5x + 8$. Choose your own starting value. Compare the overall pattern of the graphical iteration to the patterns you saw for the function in Problems 1 and 2. Make a conjecture about what kinds of linear rules yield graphical iteration patterns like the one you found in this problem. You will test your conjecture later in this lesson.

Fixed Points As you have seen, sometimes when you iterate a function, you are drawn to a particular value; and if you reach that value, you never leave it. Such a value is called a *fixed point*.

4 Consider fixed points in the case of linear functions.

a. Look back at Problem 1. What is the fixed point when iterating the function $f(x) = 0.5x + 2$?

b. What is the fixed point when iterating the function $f(x) = -0.7x + 6$?

c. The precise definition of a **fixed point** is a value x such that $f(x) = x$. Explain why this definition fits the previous description of a fixed point: "If you reach that value, you never leave it."

5 One method that sometimes works to find a fixed point is iterating the function, either numerically or graphically, and seeing what happens. Another method for finding a fixed point is to use the definition of a fixed point. Since the definition says that a fixed point is a value x such that $f(x) = x$, set the rule for the function equal to x and solve. Use this symbolic method to find the fixed point when iterating the function $f(x) = 0.5x + 2$. Compare your answer to your response in Problem 4 for Part a.

6 For each of the following functions, in Parts a–h, try to find a fixed point using each of these three methods.

Method I Iterate by using the last-answer feature of your calculator or spreadsheet software (numeric method).

Method II Iterate graphically (graphic method).

Method III Solve the equation $f(x) = x$ (symbolic method).

Organize your work as follows.

- Try a variety of starting values for each function.
- Keep a record of what you try, the results, and any patterns that you notice.
- Prepare a display of the graphical iteration.

Each student should use all three of the methods listed above for the functions in Parts a and b. For Parts c through h, share the workload among classmates.

a. $s(x) = 0.6x + 3$ **b.** $u(x) = 4.3x + 1$

c. $t(x) = 0.2x - 5$ **d.** $v(x) = 3x - 4$

e. $w(x) = x + 2$ **f.** $f(x) = -0.8x + 4$

g. $h(x) = -x + 2$ **h.** $k(x) = -2x + 5$

Slope and Long-Term Behavior Fixed points are examples of the long-term behavior of iterated functions. The different types of fixed points for linear functions can be completely characterized and predicted.

7 Three important characteristics to look for when iterating functions are *attracting fixed points*, *repelling fixed points*, and *cycles*. An **attracting fixed point** is a fixed point such that iteration sequences that start close to it get pulled into it. In contrast, iteration sequences move away from a **repelling fixed point**, except of course, for the sequence that begins at the fixed point. A **cycle** is a set of numbers in an iteration sequence that repeats over and over.

a. For each of the linear functions you iterated in Problem 6, decide with some classmates whether it has an attracting fixed point, a repelling fixed point, a cycle, or none of these.

b. Is there a connection between the slope of the graph of a linear function and the function's behavior when iterated? If so, explain how you could complete Part a of this problem simply by knowing the slope of each linear function's graph.

Summarize the Mathematics

In this investigation, you discovered that linear functions have rich and varied behavior when iterated.

a. Describe what a fixed point is and how to find one. Give as many different descriptions of how to find a fixed point as you can.

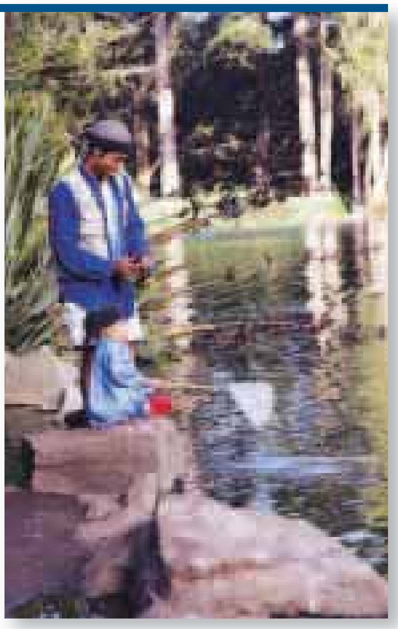
b. Attracting fixed points seem to pull you into them. But do you ever actually get to a fixed point? Explain.

c. Describe the different long-term behaviors that can occur when a linear function is iterated. For each of the behaviors described, explain how that behavior is completely characterized by the slope of the graph of the iterated function.

d. In the first lesson of this unit, you investigated situations that could be modeled by combined recursive formulas of the form $A_n = rA_{n-1} + b$. What is the connection between these combined recursive formulas and iterating linear functions?

Be prepared to share your descriptions and thinking with the entire class.

Check Your Understanding



At the beginning of this unit, you analyzed the changing fish population in a pond. The pond has an initial population of 3,000 fish. The population decreases by 20% each year due to natural causes and fish being caught. At the end of each year, 1,000 fish are added. A rule that models this situation is

$$NEXT = 0.8NOW + 1{,}000, \text{ starting at } 3{,}000.$$

a. Rewrite this rule as a recursive formula, using subscript notation.

b. What function can be iterated to produce the same sequence of population numbers generated by the recursive formula? With what value should you start the function iteration?

c. Iterate the function in Part b and describe the long-term behavior of the iteration sequence. Compare this behavior to the long-term behavior of the fish population that you discovered in Investigation 1 of Lesson 1.

d. Graphically iterate this function, starting with $x = 3{,}000$.

e. Find the fixed point by solving an equation. Is the fixed point attracting or repelling? How can you tell by examining the slope of the function's graph?

f. Explain what the fixed point and its attracting or repelling property tell you about the changing fish population.

Applications

1 Consider the three tables below. Describe similarities, differences, and connections among the three tables.

x	$f(x) = 2x + 1$
0	1
1	3
2	5
3	7
⋮	⋮

n	$U_n = 2U_{n-1} + 1$, $U_0 = 1$
0	1
1	3
2	7
3	15
⋮	⋮

x	Iterate $f(x) = 2x + 1$
0	1
1	3
3	7
7	15
⋮	⋮

2 Iterate $f(x) = \frac{1}{x}$, using several different starting values. Describe the iterated behavior. Explain why the iterated behavior makes sense because of the nature of the function f.

3 **a.** Give an example of a function and a starting value such that the iterated sequence increases without bound.

b. Give an example of a function and a starting value such that the iterated sequence oscillates between positive and negative values.

4 Experiment with iterating the function $f(x) = x^3 - x^2 + 1$.

a. Describe the behavior of the iteration sequence when you iterate $f(x)$ with the following beginning values.

i. Begin with $x = 1$.

ii. Begin with $x = 0.8$.

iii. Begin with $x = 1.2$.

b. Find a fixed point for $f(x)$. Is this fixed point repelling, attracting, some combination of repelling and attracting, or none of these? Explain.

5 Experiment with iterating $g(x) = 3.7x - 3.7x^2$.

a. Find the fixed points of $g(x)$ by writing and solving an appropriate equation.

b. Iterate $g(x)$ starting with $x = 0.74$, which is close to a fixed point. Carefully examine the iteration sequence by listing these iterations.

- List iterations 1 through 6.
- List iterations 16 through 20.
- List iterations 50 through 55.
- List iterations 64 through 68.

c. Describe any patterns you see in the iteration sequence. Does the iteration sequence get attracted to the fixed point? Does it get steadily repelled?

A fixed point is called repelling if iteration sequences that begin near it get pushed away from the fixed point at some time, even if such sequences occasionally come back close to the fixed point. Is the fixed point at about 0.73 repelling?

6 Consider iterating $h(x) = 3.2x - 0.8x^2$.

a. The fixed points of $h(x)$ are repelling. Do you think you will find them by numerical iteration? Explain your reasoning.

b. Find the fixed points of $h(x)$ using symbolic reasoning.

c. Experiment with iterating $h(x)$, starting with initial values just above and below each fixed point. Carefully describe the characteristics of each fixed point.

7 Function iteration can be used to model population change. Consider, for example, the bowhead whale population described in Applications Task 4 of Lesson 1. A status report on the bowhead whales of Alaska estimated that the 1993 population of this stock was between 6,900 and 9,200 and that the difference between births and deaths yielded an annual growth rate of about 3.1%. No hunting of bowhead whales is allowed, except that Alaskan Inuit are allowed to take, or harvest, about 50 bowhead whales each year for their livelihood. (Source: nmml.afsc.noaa.gov/CetaceanAssessment/bowhead/bmsos.htm)

a. Write a recursive formula and a corresponding function that can be iterated to model this situation.

b. Using the low population estimate as the initial value, iterate and describe the long-term behavior of the population. Do you think this model is a good one to use for predicting the bowhead whale population hundreds of years from now? Why or why not?

c. Find the fixed point. By examining the slope of the graph of the iterated function, decide if the fixed point is attracting or repelling. Illustrate your conclusions by graphically iterating the function.

d. Write a brief analysis of the changing bowhead whale population, as described by your model. As part of your analysis, describe the role played by the fixed point. Make some long-term predictions based on different initial whale populations.

Connections

8 Consider iteration of an arbitrary function, $y = f(x)$.

a. Represent the process of iterating $f(x)$ by a rule using the words *NOW* and *NEXT*.

b. Represent the process of iterating $f(x)$ by a recursive formula with subscript notation.

9 In Problem 7 of Investigation 1, you investigated the logistic equation, $f(x) = rx(1 - x)$. You found that different values of r produce different long-term iteration behavior. You can program a computer to play music that corresponds to the long-term behavior. For example, consider the BASIC program below.

```
10  volume = 50
20  duration = 0.5
30  input "r = "; r
40  x = 0.02
50  print x
60  pitch = 220*4^x
70  sound pitch,duration,volume
80  x = r*x*(1–x)
90  go to 50
```

a. Explain the purpose of each step of this program.

b. Enter this program, or an equivalent program, into a computer. Run the program using the values 2.7, 3.2, 3.5, 3.83, and 4.0 for r. Describe the results. Explain how the patterns in sound generated by the computer program compare to the numerical patterns that you found in Problem 7 of Investigation 1.

10 It is possible to use function iteration to solve equations. Consider the equation $x - 3 = 0.5x$.

a. Before solving the equation by using function iteration, solve it using at least two other methods. In each case, explain your method.

b. To solve $x - 3 = 0.5x$ by function iteration, you will make use of fixed points. Consider this equation written in an equivalent form: $x = 0.5x + 3$.

i. Explain why this form is equivalent to the original form.

ii. For what function does this equation define a fixed point?

c. Iterate the function you identified in Part b for some initial value. Describe the long-term behavior. Explain why the observed long-term behavior gives you a solution to the original equation. Compare the solution you obtained to what you found in Part a.

d. Use the method of function iteration to solve $2x + 5 = 6x - 7$. (If at first you do not succeed, try rewriting the equation in an equivalent form.)

e. Summarize the method of function iteration to solve linear equations.

f. When you know several methods for solving a given problem, you should always think about which one is "best." In the case of solving linear equations, would you rather use one of the solution methods from Part a or use function iteration?

11 In this task, you will find and use a general formula for the fixed point of a linear function.

a. Use the algebraic method for finding fixed points to derive a general formula for the fixed point of the linear function $f(x) = rx + b$.

b. Use the general formula to find the fixed point for $h(x) = 0.75x - 4$.

c. If the fixed point is attracting, then you can find it by iteration.

i. How can you tell from the formula for $h(x)$ that this function has an attracting fixed point?

ii. Iterate $h(x)$ to find the fixed point. Compare it to what you found using the general formula in Part b.

12 In Lesson 1, you investigated recursive formulas of the form $A_n = rA_{n-1} + b$. Suppose such a recursive formula represents year-to-year change in some population. Assume that r is a positive number less than 1.

a. Explain why the long-term population in this situation can be found by finding the fixed point of the function $f(x) = rx + b$.

b. Use the general fixed point formula from Part a of Connections Task 11 to explain the following patterns that you previously discovered about the fish population problem in Lesson 1.

- Changing the initial population does not change the long-term population.
- Doubling the restocking amount doubles the long-term population.
- Doubling the population-decrease rate cuts the long-term population in half.

Reflections

13. In this unit, you have studied recursion in several contexts, for example, recursive formulas and function iteration. Recursion is sometimes described as a "self-referral" process. Explain why this is a reasonable description of recursion.

14. In this lesson, you briefly explored a famous equation in mathematics and science called the logistic equation (also called the logistic map), $f(x) = rx(1 - x)$. This equation was first studied extensively in the 1970s. Some of the first discoveries about the behavior of the iterated function came from trying to apply it as a model in biology and ecology. Its behavior turned out to be surprisingly complex and profound, giving rise to what is sometimes called *chaos theory*.

   a. Review what you found out about the iterated behavior of the logistic equation in Problem 7 of Investigation 1, page 518. Why do you think the term "chaos" has been used to describe certain long-term behavior of the logistic equation?

   b. One of the first investigators of the logistic equation was an Australian physicist and biologist named Robert May. May argued that the world would be a better place if every student was given a pocket calculator and encouraged to play with the logistic equation. What do you think May meant?

   c. *Optional:* Obtain a copy of the book *Chaos: Making a New Science* by James Gleick (New York: Viking, 1987). Read Chapter 3, entitled "Life's Ups and Downs." This chapter is an entertaining account of some of the history of the logistic equation. Write a two-page report summarizing the chapter.

15. The idea of "chaos" in mathematics comes from a new area of mathematics that is sometimes called *chaos theory*. Chaos theory is related to certain long-term behavior of the logistic equation, which you examined in Problem 7 of Investigation 1, page 518. Read the article on the next page, which attempts to apply chaos theory to politics. Summarize the description of chaos given in the article.

   How does this description relate to the long-term behavior of the logistic equation? Do you think the conclusions in the article are valid? Why or why not?

Counting on Chaos to Save Day for Dole

by Al Kamen

It's come to this. Robert J. Dole's poll numbers are so bad that Rich Galen, director of political communications for House Speaker Newt Gingrich (R-Ga.), is touting "Chaos Theory" to inspire the GOP faithful.

"Stay with me, here," Galen began in a memo written last week "For Distribution to Talk Show Hosts," a regular salvo he sends out to about 100 or so conservative radio folks.

"There is a relatively new branch of science which is called Chaos Theory," he explained. It talks about a "butterfly fluttering its wings in Argentina which ultimately leads to a thunderstorm in New Jersey."

But "you will not be able to predict, with any degree of precision, when lightning will form and strike … One second there is no lightning, and the next second the sky is bright. Chaos."

He went on: "Take another example. Suppose you take a wineglass and begin to squeeze it at its upper rim. If you continue to apply pressure, at some point the glass will break. The system will collapse entirely and instantaneously. Until the moment it breaks, it will be a perfectly usable glass. After the glass breaks, it will be nothing but a pile of shards."

"What does this have to do with the presidential campaign?" Galen asked, which seems like a pretty good question.

"My strong impression is there will come a time … when the Clinton campaign, like the glass, will entirely and instantaneously collapse. One moment it will be a campaign, the next moment it will be unrecognizable."

"That's why we don't have to be frightened by the current Dole-Clinton polling numbers," he said. Chaos theory will save the day, or at least win New Jersey.

"What we must do, however, is to continue to keep the pressure on. If we get discouraged [and] stop squeezing the rim of the glass, then the glass will never break."

Now we know why the Republicans are infinitely more interesting than the Democrats. The Republicans look to science. All the Democrats can say is: "It's the economy, stupid."

Source: *The Washington Post*, September 16, 1996.

16 Explain why the fixed points for a function $f(x)$ correspond to the points of intersection of the graphs of $y = f(x)$ and $y = x$.

17 Why do you think it is sometimes said that you can never "see" a repelling fixed point?

Extensions

18 The recursive formula for a geometric sequence looks like a combined recursive formula of the form $A_n = rA_{n-1} + b$ without the added b. This connection can be used to find a function formula for such combined recursive formulas. The strategy involves building from a function formula for a geometric sequence.

a. What is the function formula for a geometric sequence with recursive formula $A_n = rA_{n-1}$ and initial value A_0?

b. As the next step, think about the long-term behavior of $A_n = r^nA_0$. Compare it to the long-term behavior of $A_n = rA_{n-1} + b$. Consider the situation when $|r| < 1$.

- **i.** Explain why $A_n = rA_{n-1} + b$ has an attractive fixed point for its long-term behavior.
- **ii.** What is the long-term behavior of $A_n = r^nA_0$?

c. Now modify the formula $A_n = r^nA_0$ so that it has the same long-term behavior as $A_n = rA_{n-1} + b$, as follows. Begin by adding the fixed point, denoted *FIX*, to the function formula $A_n = r^nA_0$ so that the new function will have the same long-term behavior as $A_n = rA_{n-1} + b$. Explain why $A_n = r^nA_0 + FIX$ has long-term behavior converging to *FIX*, if $|r| < 1$.

d. Finally, modify the formula $A_n = r^nA_0 + FIX$ so that it has the same initial value A_0 as $A_n = rA_{n-1} + b$.

- **i.** Explain why the initial value of $A_n = r^nA_0 + FIX$ is equal to $A_0 + FIX$.
- **ii.** Explain why the initial value of $A_n = r^n(A_0 - FIX) + FIX$ is equal to A_0.

e. As you explained in Part d, the function formula $A_n = r^n(A_0 - FIX) + FIX$ has the same long-term behavior and the same initial value as the combined recursive formula $A_n = rA_{n-1} + b$. Use this new type of formula to find A_5 for $A_n = 2A_{n-1} + 1$, with $A_0 = 3$. Then compute A_5 by successively evaluating $A_n = 2A_{n-1} + 1$, and compare your two results.

f. Compare the function formula in Part e to the function formula you derived in Connections Task 11 of Lesson 2 (page 503). Resolve any apparent differences.

19 In Connections Task 10, you investigated the method of function iteration to solve linear equations. Investigate if this method will work for quadratic equations. Consider the quadratic equation $2x^2 + 5x = 3$.

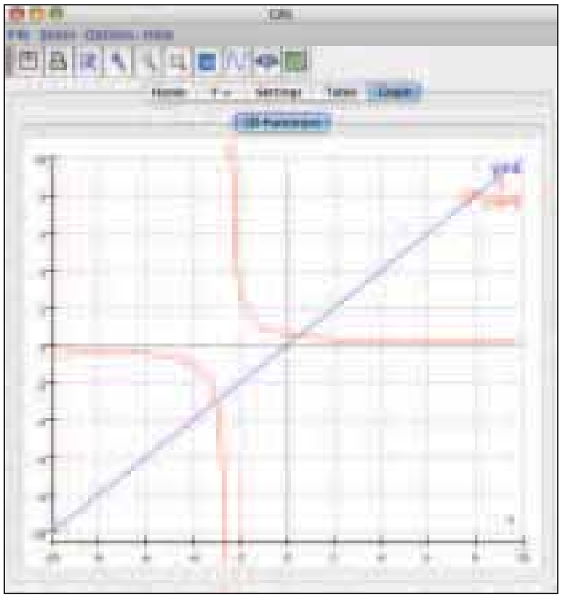
a. Find a function that you could iterate in order to use the method of function iteration to solve this equation. Iterate with several different initial values. Describe the results in each case.

b. Solve this equation using another method. How many solutions are there?

c. What properties of fixed points allow you to either find or not find a solution when using the method of function iteration?

d. Use your calculator or computer software to help you sketch graphs of the iterated function and $y = x$ on the same set of axes. (Your graph may look different than the one shown here, depending on which function you found to iterate in Part a.) Locate the fixed points on this graph. At each of the fixed points, visualize a line drawn tangent to the graph of the iterated function at the fixed point.

CPMP-Tools

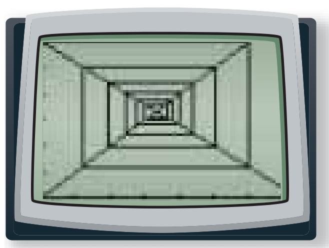


Estimate the slope of each of these tangent lines. For which fixed point is the absolute value of the slope of the tangent line greater than 1? For which tangent line is the absolute value of the slope less than 1?

e. Recall the connection between iterating linear functions and slope. Explain how the slope of the tangent line at a fixed point tells you if the fixed point is attracting or repelling. Explain what attracting or repelling fixed points have to do with solving nonlinear equations using the method of function iteration.

20 Although it is relatively easy to iterate linear functions graphically by hand with a fair degree of accuracy, it is quite difficult to iterate nonlinear functions graphically. This is because, as you have seen, the shape of a graphical iteration is determined by the slope of the graph of the iterated function. And while lines have constant slopes, graphs of nonlinear functions have changing slopes. Thus, it is usually necessary to use a computer or graphing calculator to accurately iterate nonlinear functions graphically. Graphical iteration capability is built into many calculators and computer graphing packages.

a. Consult a manual, if necessary, to find how to use your calculator or computer software to iterate graphically. Practice by graphically iterating $f(x) = -0.8x + 6$. You should get a graph that looks like the one at the right.

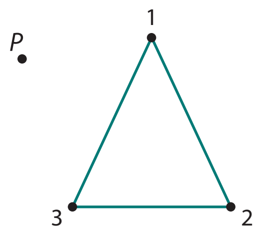


b. In Applications Task 6, you were asked to iterate the function $h(x) = 3.2x - 0.8x^2$. Use a graphing calculator or computer software to iterate $h(x)$ graphically, starting with $x = 2.6$. Compare the graphical iteration pattern to the numerical iteration results.

c. In Applications Task 4, you were asked to iterate the function $f(x) = x^3 - x^2 + 1$. Graphically iterate $f(x)$ to illustrate each of your results in Applications Task 4.

21 In this lesson, you iterated algebraic functions. It is also possible to iterate geometric transformations. As an example, play the following "Chaos Game." (Algorithm originally described in Barnsley, M., *Fractals Everywhere*, Academic Press, 1988.)

a. On a clean sheet of paper, draw the vertices of a large triangle. Any type of triangle will work; but for your first time playing the game, use an isosceles triangle. Label the vertices with the numbers 1, 2, and 3.



b. Start with a point anywhere on the sheet of paper. This is your initial input. Randomly choose one of the vertices (for example, use a random number generator to choose one of the numbers 1, 2, or 3). Mark a new point one-half of the distance between your input and that vertex. This is your first output and also your new input. Then randomly choose another vertex. Mark the next point, half the distance from the new input to that vertex. Repeat this process until you have plotted six points.

c. The goal of the Chaos Game is to see what happens in the long term. What do you think the pattern of plotted points will look like if you plot 300 points? Make a conjecture.

d. Program a calculator or computer to play the Chaos Game, or find such a program on the Internet, and then carry out several hundred iterations. Since you are interested only in the long-term behavior, you might carry out the first ten iterations without plotting the resulting points and then plot all points thereafter.

e. Repeat the game for several other initial points. Do you think you will always get the same resulting figure? Try it.

f. The figure that results from the Chaos Game is an example of a familiar *fractal*. One of the most important characteristics of fractals is that they are *self-similar*, which means that if you zoom in, you keep seeing figures just like the original figure. What is the scale factor of successively smaller triangles in the fractal that you produced?

g. Give a geometrical explanation for why the Chaos Game will always generate a Sierpinski triangle. (For more about Sierpinski triangles, see Applications Task 3 from Lesson 2 on page 500.)

Review

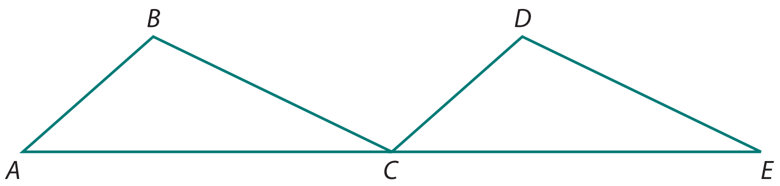
22 Find the zeroes of each function.

a. $f(x) = (x - 5)(x + 3)(2x - 1)$

b. $g(x) = (x^2 - 7x + 12)(x - 6)$

c. $h(x) = x^2 + 7x + 3$

23 In the figure below, $\overline{AB} \parallel \overline{CD}$, $\overline{BC} \parallel \overline{DE}$, and C is the midpoint of $\overline{AE}$. Prove that $\triangle ABC \cong \triangle CDE$.



24 Use symbolic reasoning to determine if each statement is true or false.

a. $(x + 4)^2 - 3 = x^2 + 13$

b. $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$

c. $3(3^x)(9^x) = 9^{2x}$

d. $\sqrt{a^2 + b^2} = a + b$

25 Sketch a graph of each function. Then state the domain and range of the function.

a. $f(x) = -\frac{7}{3}x - 5$

b. $g(x) = x^2 + 6$

c. $h(x) = \frac{6}{x}$

d. $j(x) = 3(2^x)$

e. $k(x) = |x| - 2$

f. $p(x) = |x + 3|$

26 If a process is in control and data collected from the process are approximately normally distributed, what is the probability that the next reading on a control chart lies outside the control limits?

27 Find each product without using your calculator.

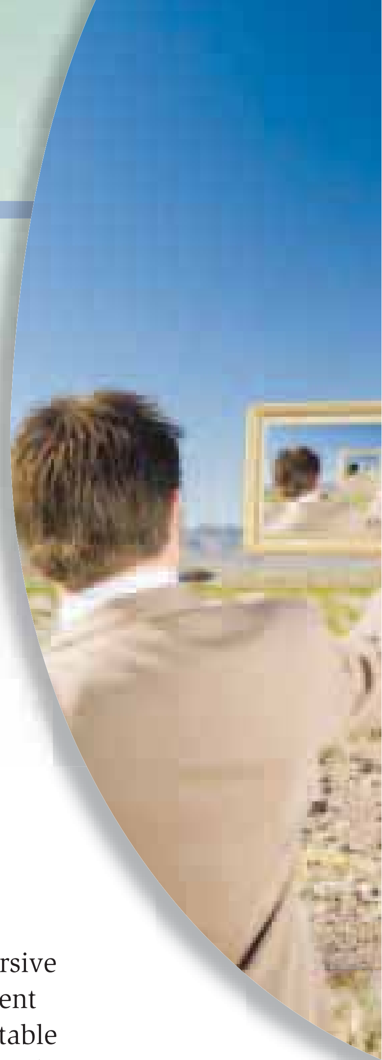
a. $\begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 1 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

LESSON 4

Looking Back



In this unit, you have investigated sequential change in a variety of contexts using the tools of recursion and iteration. You have extended the idea of using *NOW-NEXT* formulas to model sequential change; you have studied recursive formulas, function iteration, and sequences; and you have made connections to previous work with linear, exponential, and polynomial functions. In this final lesson, you will pull together and review the key ideas in the unit.

1 In Lesson 1, you modeled a variety of situations with combined recursive formulas of the form $A_n = rA_{n-1} + b$. In Lesson 3, you iterated linear functions. These two topics are closely connected. In this task, you will summarize key features of that connection.

By using different values for r and b in the combined recursive formula $A_n = rA_{n-1} + b$, you can build models for different situations. Four different possibilities are indicated by the table below. One recursive formula has already been entered into the table. If you completed Applications Task 6 on page 471, you already have a start on this task.

Four Different Versions of the Recursive Formula $A_n = rA_{n-1} + b$

	$0 < r < 1$	$r > 1$
$b < 0$		
$b > 0$	$A_n = 0.8A_{n-1} + 1{,}000$	

Choose *one* of the empty table cells. In a copy of the table, write an appropriate recursive formula in the cell and then analyze the recursive formula and its use as a mathematical model. Title your work with the particular recursive formula on which you are reporting. Organize your analysis of the recursive formula as follows.

a. Briefly describe a real-world situation that can be modeled by the recursive formula along with a chosen initial value.

b. Rewrite the recursive formula as a *NOW-NEXT* formula.

c. Write a linear function that can be iterated to yield the same sequence as the successive values of the recursive formula. Choose an initial value.

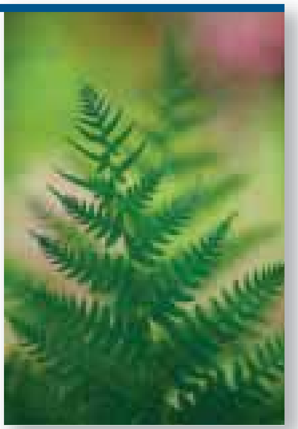
  i. Iterate the function, and describe the long-term behavior.

  ii. Find the fixed point. Decide whether it is attracting, repelling, or neither. Explain in terms of slope.

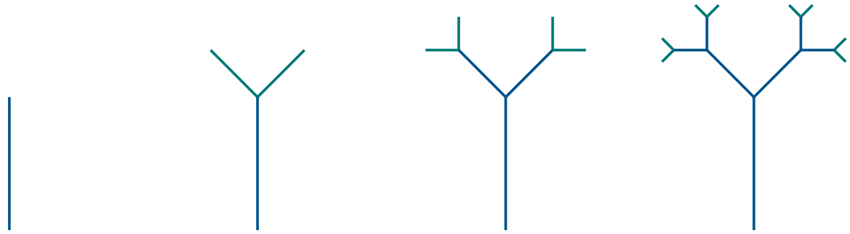
  iii. Sketch a graph showing graphical iteration of the function for the initial value previously chosen.

d. Sketch a graph of A_n versus n, using the same initial value that you chose in Part c.

e. Describe the long-term behavior of the real-world situation being modeled, for different initial values. Refer to the fixed point and its properties, but keep your description in the context of the particular situation being modeled.



2 Many irregular shapes found in the natural world can be modeled by fractals. Study the first few stages of the fractal tree shown below.



Stage 0 Stage 1 Stage 2 Stage 3

a. Write the number of new branches at each stage for the first several stages. Then write recursive and function formulas that describe this sequence. Use one of the formulas to predict the number of *new* branches at Stage 12. Check your prediction using the other formula.

b. Find the *total* number of branches at Stage 12.

c. Suppose that the length of the initial branch is 1 unit and that the branches at each successive stage of the fractal tree are half the length of the branches at the previous stage.

  i. Write the total length of all the branches at each stage for the first several stages.

  ii. Find the total length of all the branches at Stage 15.

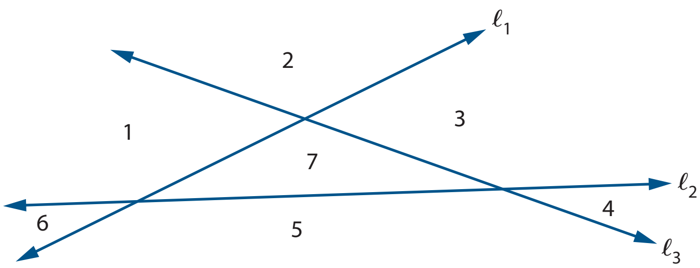
3 When you attend a movie, concert, or theater production, you may notice that the number of seats in a row increases as you move from the front of the theater to the back. This permits the seats in consecutive rows to be offset from one another so that you have a less-obstructed view of the stage.

The center section of the orchestra level of Shaw Auditorium is arranged so that there are 42 seats in the first row, 44 seats in the second row, 46 seats in the third row, and so on for a total of 25 rows.

a. Determine the number of seats in the last row in two different ways. Compare your result and your methods to those of other students. Resolve any differences.

b. Determine the total number of seats in the center section of the orchestra level in at least two different ways. One method should involve using a rule showing the total number of seats as a function of number of rows n.

4 Amy was investigating the maximum number of regions into which a plane is separated by n lines, no two of which are parallel and no three of which intersect at a common point. For example, the diagram below shows the maximum number of regions for 3 lines.



Amy gathered the data shown in the table below.

Number of Lines	0	1	2	3	4
Number of Regions	1	2	4	7	11

a. Verify the entries in the table.

b. Find a function formula for the maximum number of regions formed by n lines, using the method of finite differences.

c. Write a recursive formula for this relationship.

5 Find recursive and function formulas for each of the sequences below. Then find the sum of the terms up through the term with subscript 15.

a.

n	0	1	2	3	4	5	6	...	15	...
L_n	600	300	150	75	37.5			...		...

b.

n	1	2	3	4	5	6	7	...	15	...
P_n	−3	2	7	12	17			...		...

Summarize the Mathematics

When asked to make a list of topics covered in this unit, Derartu produced the following list.

- Recursive formulas
- Population growth
- Compound interest
- Arithmetic sequences and sums
- Geometric sequences and sums
- Finite differences
- Linear functions
- Exponential functions
- Polynomial functions
- Function iteration
- Fixed points

a Examine each topic to see if you agree that it should be on the list. Discuss any questions that you have about what a particular topic is or why it is on the list. Add any other topics that you think are missing.

b Ray's list had just one item on it: combined recursive formulas. Explain why Ray's list provides a reasonable summary of the unit. Which of the topics on Derartu's list are connected in some way to combined recursive formulas of the form $A_n = rA_{n-1} + b$?

Be prepared to share your analysis of the unit topics with the entire class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.