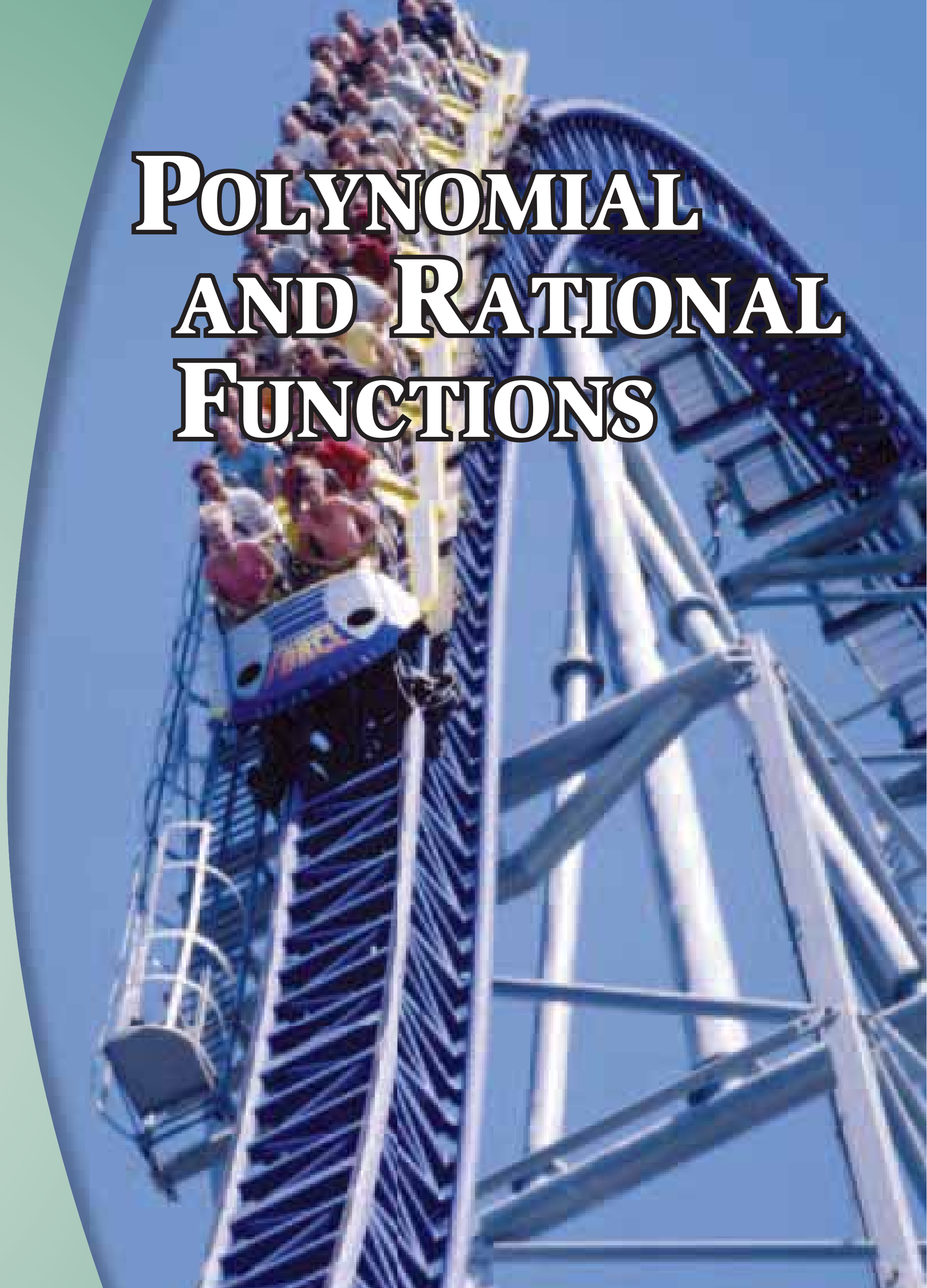
UNIT 5

POLYNOMIAL AND RATIONAL FUNCTIONS



In prior units of *Core-Plus Mathematics*, you developed understanding and skill in the use of linear, quadratic, and inverse variation functions. Those functions are members of two larger families of versatile and useful algebraic tools, the polynomial and rational functions.

In this unit, you will learn how to use polynomial and rational expressions and functions to represent and analyze a variety of quantitative patterns and relationships. The key ideas will be developed through work on problems in three lessons.

Lessons

1 *Polynomial Expressions and Functions*

Use linear, quadratic, cubic, and quartic polynomial functions to represent quantitative relationships, data patterns, and graphs. Analyze the connections between symbolic expressions and graphs of polynomial functions. Combine polynomials by addition, subtraction, and multiplication.

2 *Quadratic Polynomials*

Extend skills in expanding and factoring quadratic expressions. Use a completing-the-square strategy to write quadratic expressions in equivalent forms and to prove the quadratic formula for solving quadratic equations.

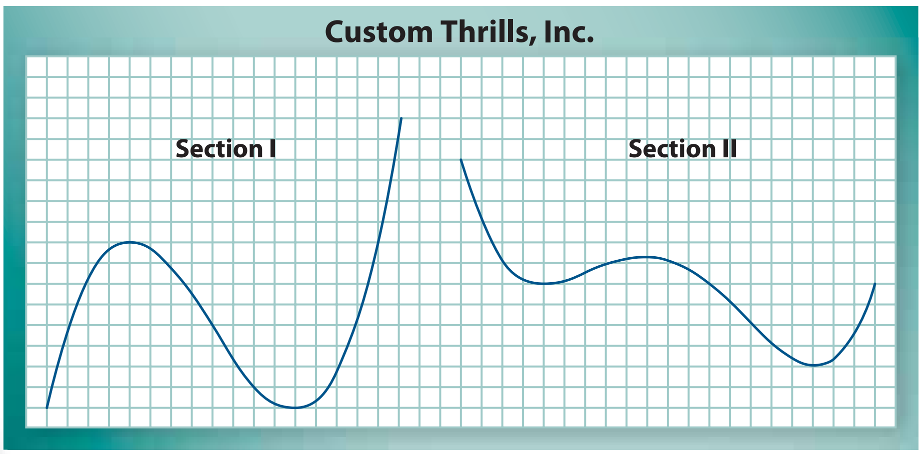
3 *Rational Expressions and Functions*

Use quotients of polynomial functions to represent and analyze quantitative relationships. Analyze the connections between symbolic expressions and graphs of rational functions. Combine rational expressions by addition, subtraction, multiplication, and division.

LESSON 1

Polynomial Expressions and Functions

Working for a company that designs, builds, and tests rides for amusement parks can be both fun and financially rewarding. Suppose that in such a position, your team is in charge of designing a long roller coaster. One morning, your team is handed sketches that show ideas for two sections of a new roller coaster.



As team leader, your task is to find algebraic functions with graphs that match the two sketches. The functions will be useful in checking safety features of the design, like estimated speed and height at various points of the track. They will also be essential in planning manufacture of the coaster track and support frame.

Think About This Situation

Study the sketches of Section I and Section II of the proposed coaster design.

a) What familiar functions have graphs that match all or parts of the design sketches?

b) What strategies could you use to find functions with graphs that model the sketches?

c) What do you think are the key points on each sketch that should be used in finding a function model for the graph pattern?

In this lesson, you will begin study of an important class of algebraic functions called *polynomial functions*. You will learn how to use polynomial functions to model complex graphical patterns, like the roller coaster design proposals, and how to use operations on polynomials to solve problems in business situations.

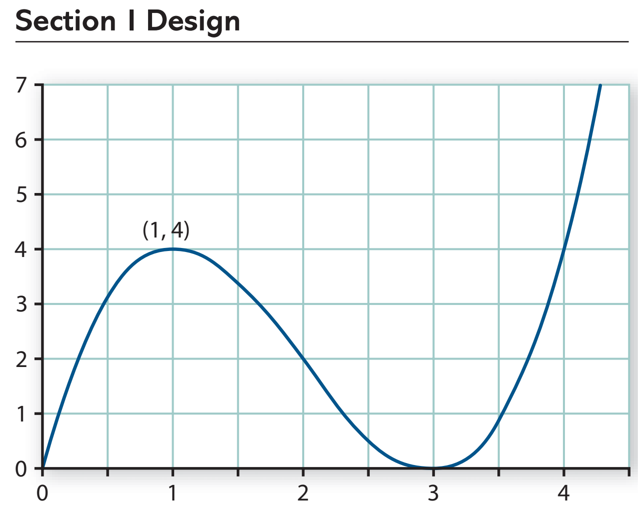
Investigation 1 Modeling with Polynomial Functions

The roller coaster design sketches are not modeled well by graphs of linear, exponential, or quadratic functions. But there are other kinds of algebraic functions that do have such graphs, and there are techniques for using sample graph points to find rules for those functions. In this investigation, you should look for answers to these questions:

What are polynomial functions and what kinds of graphs do those functions have?

How are rules for polynomial functions related to patterns in their graphs?

To the *n*th degree The following diagram shows the graph of a function that could represent a roller coaster design based on the Section I sketch.



1. When you used curve-fitting tools in earlier work, all you had to do was supply coordinates of points that capture key features of the graph or data pattern and then select a function type for the model. Why would neither linear, quadratic, exponential, nor inverse variation functions provide good models for the graph pattern representing the Section I Design?

2. You might have noticed that most curve-fitting tools offer other modeling options. The one listed right after the quadratic option is usually *cubic*.

   a. What key points on the graph, beside (1, 4), do you think would be helpful in finding a cubic function that models the proposed Section I Design?

   b. Using the points you selected in Part a, apply a cubic curve-fitting routine to find a function model for the graph pattern.

   c. Compare the graph of the resulting cubic function to the shape of the Section I Design. Describe ways that the cubic function is or is not a good model of that pattern.

3. Compare results of your work toward a function model for the Section I Design with those of others in your class.

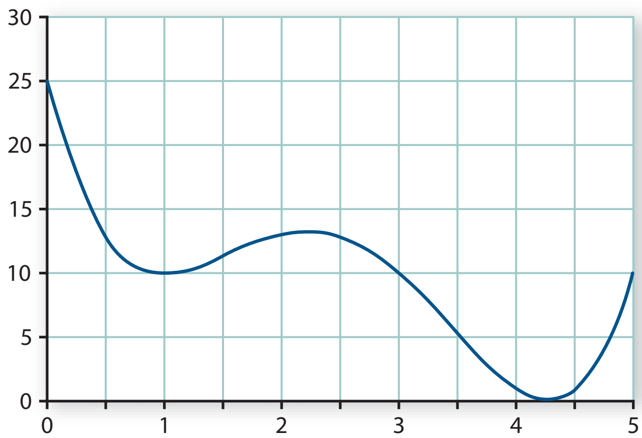
   a. Did everyone use the same points from the given graph?

   b. For those who used different points or even different numbers of points, were the resulting function models still the same?

   c. What do you think might be the minimum number of points needed to find a cubic model for a data or graph pattern?

4 As you may have noted, there are some significant differences between the design ideas for Sections I and II of the proposed roller coaster. The next diagram shows a graph of a function that could represent a roller coaster design based on the Section II sketch.

Section II Design



a. Find coordinates of key points outlining the shape of this graph. Then find the cubic function model for the pattern in those points and compare its graph to the shape of the proposed Section II Design.

b. You may have noticed there are other curve-fitting options on your calculator or computer software list. One such option is *quartic*. What do you think might be the minimum number of points needed to find a quartic model for a data or graph pattern?

c. Find a quartic function model for the pattern of data points you identified in Part a. Check how well its graph matches the pattern in the Section II Design.

d. Compare your result with those of others in your class. See if you can explain any differences.

Though you may not have been aware of it at the time, in earlier *Core-Plus Mathematics* units involving linear and quadratic functions, you were learning about functions from a larger class called polynomial functions. Cubic and quartic functions, like those that you explored in a search for models of the roller coaster designs, are also polynomial functions. A **polynomial function** is any function with a rule that can be written in the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is a whole number and the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are numbers.

The functions described by information on these calculator screens are cubic and quartic polynomials.

Any algebraic expression in the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is called a **polynomial expression**. One of the most important characteristics of any polynomial function or expression is its degree. The **degree of a polynomial** is the greatest exponent of the variable that occurs in the expression. For example, a quadratic polynomial has degree 2, a cubic polynomial has degree 3, and a quartic polynomial has degree 4. A nonzero constant is a polynomial of degree 0.

Connecting Polynomial Expressions and Graphs From earlier work with linear, quadratic, exponential, and power functions, you know that it is often helpful to inspect a function rule and estimate the shape of its graph. In the other direction, it is helpful if you can inspect a graph and predict the kind of function rule that will model its pattern.

5 As you have seen in exploring polynomial models for roller coaster designs, one of the most interesting and important characteristics of polynomial functions is the number and location of *peaks* or *valleys* in their graphs. A peak indicates a *local maximum* value for the function, and a valley indicates a *local minimum* value for the function.

a. Graph and then estimate coordinates of local maximum and local minimum points for each polynomial function.

i. $g(x) = -x^3 + 6x^2 - 9x$

ii. $h(x) = -x^4 + 2x^3 + 7x^2 - 8x - 12$

b. How do the examples in Part a help to explain why the adjective “local” is used in describing function values at peaks and valleys of polynomial functions?

c. Why are the points (0, 0) and (4.25, 7) *not* considered local minimum or local maximum points for the cubic function you found in Problem 2 to model the Section I Design for the roller coaster?

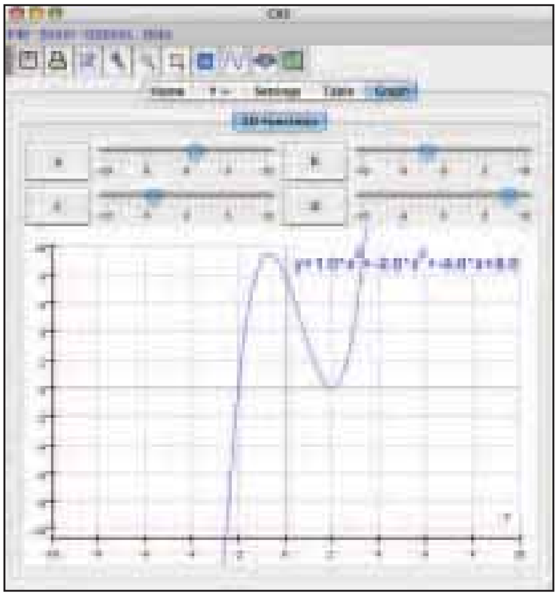
d. Why are the points (0, 25) and (5, 10) *not* considered local maximum or local minimum points for the quartic function you found in Problem 4 to model the Section II Design?

6 Consider the relationship between the degree of a polynomial function and the number of local maximum and/or local minimum points on the graph of that function.

a. Give an example of a polynomial function with no local maximum or local minimum point.

b. How many local maximum and/or local minimum points can there be on the graph of a quadratic polynomial?

7 To explore the graphs of higher degree polynomial functions, it is helpful to use computer software that accepts function definitions like $f(x) = ax^3 + bx^2 + cx + d$, allows you to move "sliders" controlling the values of a, b, c, and d, and quickly produces corresponding graphs.



CPMP-Tools

a. How many local maximum and/or local minimum points do you think there can be on the graph of a cubic polynomial function?

b. Test your conjecture in Part a by graphing cubic polynomial functions $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ for various sets of values for the coefficients a_3, a_2, a_1, and a_0. You might start with examples like these.

$$y = x^3 - 9x$$
$$y = x^3 - 6x^2 + 12x - 3$$
$$y = -x^3 + 4x + 2$$

Then modify those examples to test other cases.

c. How many local maximum and/or local minimum points do you think there can be on the graph of a quartic polynomial?

d. Test your conjecture in Part c by graphing quartic polynomial functions $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ for various combinations of values for the coefficients a_4, a_3, a_2, a_1, and a_0. You might start with examples like these.

$$y = x^4 - 9x^2 + 2$$
$$y = -x^4 - 8x^3 - 32x^2 - 32x + 5$$
$$y = -x^4 - 4x^3 + 4x^2 + 16x$$

Then modify those examples to test other cases.

e. What seems to be the connection between the degree of a polynomial and the number of local maximum and/or local minimum points on its graph? Test your conjecture by exploring graphs or tables of values for some polynomial functions of even higher degree like $p(x) = x^5 - 5x^3 + 4x$ and $q(x) = x^5 + x^3 + x + 4$.

Summarize the Mathematics

In this investigation, you explored some basic properties and uses of polynomial functions and expressions.

- **a** What kinds of expressions and functions are called polynomials?
- **b** What is the degree of a polynomial?
- **c** What are local maximum and local minimum points of a graph? How is the number of such points on the graph of a polynomial function related to the degree of the polynomial expression used in its rule?
- **d** How many points are needed for use of a curve-fitting tool to find the polynomial function of degree n that fits a data or graph pattern?

Be prepared to explain your ideas to the class.

Check Your Understanding

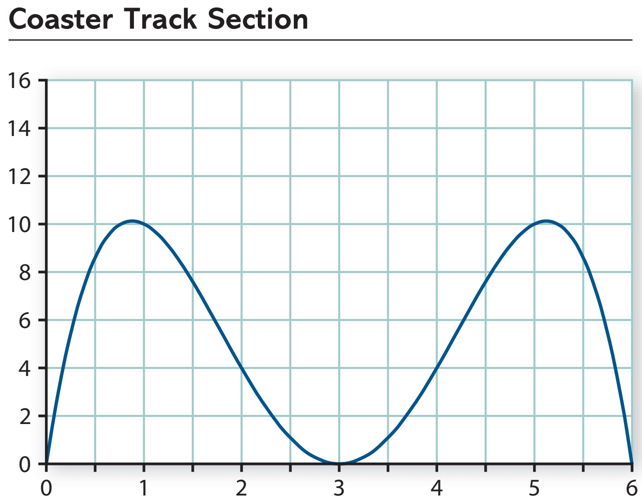
Use what you have learned about polynomial functions to help complete the following tasks.

a. Consider the functions $f(x) = 2x^3 + 8x^2 + 3x - 2$ and $g(x) = x^4 - 8x^3 + 16x^2 + 4$.

  i. Identify the degree of each function.

  ii. Estimate coordinates of the local maximum and/or local minimum points on graphs of the functions.

b. Find a function whose graph models the following coaster track design.



Investigation 2 Addition, Subtraction, and Zeroes

From early in your mathematical studies, you have learned how to perform arithmetic operations on numbers. As it turns out, you can do just about any operation with algebraic expressions that you can do with numbers. In this investigation, you will begin exploration of arithmetic operations on polynomials. As you work through the problems, look for answers to these questions:

How can the rules for polynomial functions $f(x)$ and $g(x)$ be combined to give rules for $f(x) + g(x)$ and $f(x) - g(x)$?

How are the degrees of expressions being added or subtracted related to the degree of the result?

How is the degree of a polynomial related to the number of zeroes for the function?

Adding and Subtracting Polynomials When a small music venue books a popular band, like *Ice and Fire*, business prospects of the event depend on how the ticket prices are set. For example, if the ticket price is set at x dollars, income and expenses might be estimated as follows.

Ticket sale income:
$t(x) = -25x^2 + 750x$

Snack bar income:
$s(x) = 7{,}500 - 250x$

Concert operating expense:
$c(x) = 4{,}750 - 125x$

Snack bar operating expense:
$b(x) = 2{,}250 - 75x$

1. Before each show, the manager uses the functions $t(x)$ and $s(x)$ to estimate total income.

   a. Why does it make sense that each source of income—ticket sales and snack bar sales—might depend on the price set for tickets?

   b. What income should the manager expect from ticket sales alone if the ticket price is set at $12? What income from snack bar sales? What income from the two sources combined?

   c. What rule would define the function $I(x)$ that shows how combined income from ticket sales and snack bar sales depends on ticket price? Write a rule for $I(x)$ that is in simplest form for calculation.

   d. How does the degree of $I(x)$ compare to the degrees of $t(x)$ and $s(x)$?

2. The manager also uses the functions $c(x)$ and $b(x)$ to estimate total operating expenses.

   a. Why does it make sense that each source of expense—concert operation and snack bar operation—might depend on the price set for tickets?

   b. What expense should the manager expect from concert operations alone if the ticket price is set at $12? What expense from snack bar operations? What expense from the two sources combined?

   c. What rule would define the function $E(x)$ that shows how combined expense from concert and snack bar operations depends on ticket price? Write a rule for $E(x)$ that is in simplest form for calculation.

   d. How does the degree of $E(x)$ compare to the degrees of $c(x)$ and $b(x)$?

3. Consider next the function $P(x)$ defined as $P(x) = I(x) - E(x)$.

   a. What does $P(x)$ tell about business prospects for the music venue?

   b. Write two equivalent rules for $P(x)$.

   - one that shows the separate expressions for income and operating expenses
   - another that is in simplest form for calculation

   Check your work with a CAS and resolve any differences in the results.

   c. How does the degree of $P(x)$ compare to that of $I(x)$ and of $E(x)$?

   d. Compare $E(x) - I(x)$ to $I(x) - E(x)$. What caution does this result suggest in using subtraction to find the difference of quantities represented by polynomial functions?

4 In your analysis of business prospects for a concert by *Ice and Fire*, you used common sense to combine polynomials by addition and subtraction. You probably noticed a connection between the degree of a sum or difference of polynomials and the degrees of the polynomials being combined.

a. Test your ideas about the degree of the sum or difference of polynomials by finding the simplest rules for the sum and difference of each pair of functions given below. Compare your results with those of your classmates and resolve any differences.

i. $f(x) = 3x^3 + 5x - 7$ and $g(x) = 4x^3 - 2x^2 + 4x + 3$

ii. $f(x) = 3x^3 + 4x^2 + 5$ and $g(x) = -3x^3 - 2x^2 + 5x$

iii. $f(x) = x^4 + 5x^3 - 7x + 5$ and $g(x) = 4x^3 - 2x^2 + 5x + 3$

iv. $f(x) = x^4 + 5x^3 - 7x + 5$ and $g(x) = 4x^3 + 5x$

v. $f(x) = 6x^4 + 5x^3 - 7x + 5$ and $g(x) = 6x^4 + 5x$

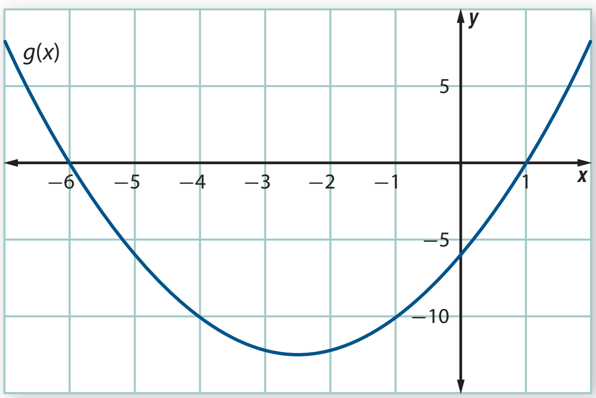
b. Explain in your own words.

i. how to find the simplest rule for the sum or difference of two polynomials

ii. how the degree of the sum or difference of two polynomials is related to the degrees of the polynomials being combined

Zeroes of Polynomial Functions In your previous work with linear and quadratic polynomial functions, you discovered that answers to many important questions are found by solving equations. You also discovered that linear and quadratic equations can be transformed to equivalent forms that look like $a_1x + a_0 = 0$ or $a_2x^2 + a_1x + a_0 = 0$, respectively.

Solving these equations requires finding values of x that are called **zeroes** of the related functions $f(x) = a_1x + a_0$ and $g(x) = a_2x^2 + a_1x + a_0$. For example, the zeroes of $g(x) = x^2 + 5x - 6$ are -6 and 1 because $g(-6) = 0$ and $g(1) = 0$. The zeroes locate the x-intercepts of the graphs of the functions.



The next problems explore the possibilities for zeroes of higher degree polynomial functions.

5 The functions $f(x) = x^3 - 6x^2 + 9x$ and $g(x) = x^4 - 10x^3 + 32x^2 - 38x + 25$ could be used to model patterns in the roller coaster designs at the beginning of this lesson.

a. Graph and find the zeroes of $f(x)$.

b. Graph and find the zeroes of $g(x)$.

c. What might the zeroes represent in the roller coaster scenario?

6 Now consider possible connections between the degree of a polynomial and the number of zeroes of the related function. It might be helpful to use function graphing software that provides "sliders" to adjust coefficients and show the corresponding graphs.

CPMP-Tools

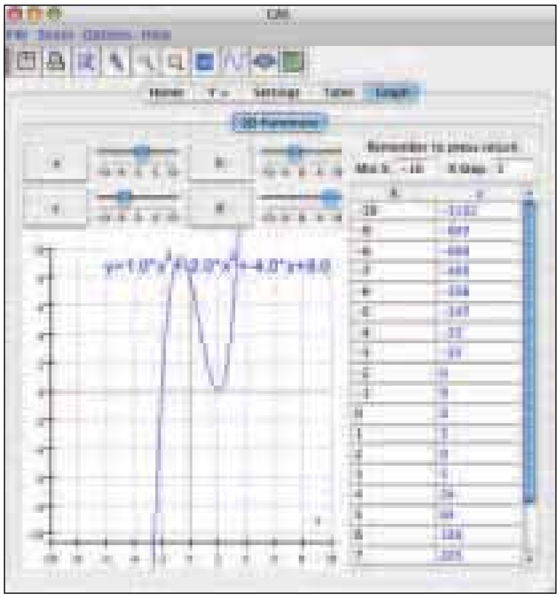
a. What are the possible numbers of zeroes for linear functions? Make sketches that represent your ideas.

b. What are the possible numbers of zeroes for quadratic functions? Make sketches that represent your ideas.

c. Explore tables and graphs of several functions with rules in the form

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \quad (a_3 \neq 0)$$

to discover the possible numbers of zeroes for cubic polynomial functions. Make sketches of results that illustrate your ideas.



d. Explore tables and graphs of functions with rules in the form

$$g(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \quad (a_4 \neq 0)$$

to discover the possible numbers of zeroes for quartic polynomials. Make sketches of results that illustrate your ideas.

e. How does the degree of a polynomial seem to be related to the number of zeroes of the related polynomial function? Test your conjecture by studying graphs of some polynomial functions of degrees five and six.

Remember: When checking to see if a mathematical idea is correct, you should explore many different examples to see how, if at all, some example might be constructed that disproves the conjecture. Here again, it might be helpful to use computer software that accepts function definitions like $f(x) = ax^3 + bx^2 + cx + d$ and allows you to move "sliders" controlling the values of a, b, c, and d to quickly produce corresponding graphs.

Summarize the Mathematics

In this investigation, you explored addition and subtraction of polynomials and zeroes of polynomial functions.

a What steps should one follow in finding a simplest rule for the sum or difference of two polynomials?

b How is the degree of the sum or difference of two polynomials related to the degrees of the polynomials being combined?

c What does the degree of a polynomial tell you about the possible number of zeroes for the corresponding polynomial function?

Be prepared to explain your ideas to the class.

Check Your Understanding

Consider the following polynomial functions.

$$f(x) = x^3 - 16x$$
$$g(x) = 16x^2 - 8x^3 + x^4$$
$$h(x) = x^4 - 8x^3 + 16x^2 + 4$$

a. For each function, (1) identify its degree, (2) sketch its graph, and (3) find its zeroes.

b. Find expressions in standard polynomial form for $f(x) + g(x)$ and $f(x) - g(x)$.

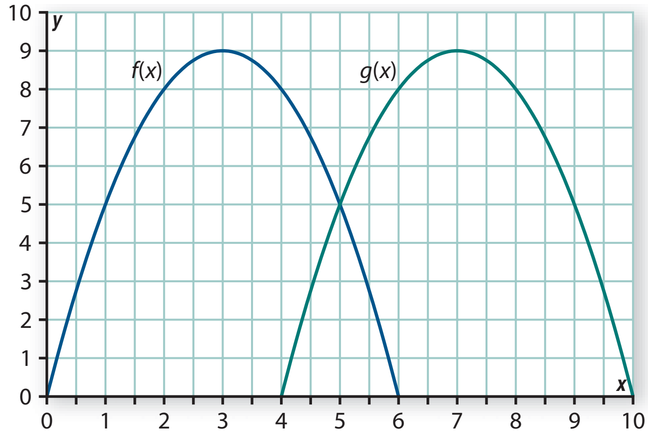
c. Find the degrees of these polynomials.

i. $g(x) - h(x)$

ii. $h(x) - f(x)$

Investigation 3 Zeroes and Products of Polynomials

In earlier work with quadratic functions, you saw how both factored and expanded expressions for function rules provide useful information. For example, the two parabolas that make up the "M" in the following diagram can be created by graphing $f(x) = -x^2 + 6x$ and $g(x) = -x^2 + 14x - 40$.



You have learned that the rules for those functions can be used to determine the location of maximum points, lines of symmetry, and y-intercepts of the graphs.

The function rules can be expressed in equivalent factored forms $f(x) = -x(x - 6)$ and $g(x) = -(x - 4)(x - 10)$. Those forms easily reveal the x-intercepts of the graphs.

Your work in this investigation will reveal strategies for working with factors and products of other polynomials. Look for answers to these questions:

How are the zeroes of a polynomial function related to the zeroes of its factors?

How can a product of polynomial factors be expanded to standard form?

How is the degree of a product of polynomials related to the degrees of the factors?

1 Analyze the following quadratic functions by using algebraic reasoning with the standard polynomial and factored forms of their rules.

a. $h(x) = x^2 + 4x$

b. $j(n) = -n^2 + n + 6$

c. $k(x) = (2x - 1)(x + 5)$

For each function:

i. write the rule in both standard polynomial and factored form.

ii. show how to use the factored form to find zeroes of the function and x-intercepts of its graph.

iii. show how to use the factored form and information about the x-intercepts to find the line of symmetry, the maximum or minimum point, and the y-intercept of the graph.

iv. show how to use the standard polynomial form to locate the line of symmetry, maximum or minimum point, and y-intercept of the graph.

2 Next, consider the function $q(x) = x(x - 3)(x + 5)$.

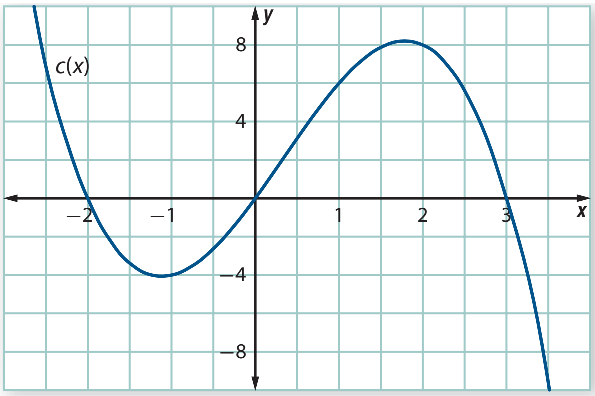
a. What are the zeroes of $q(x)$?

b. Use reasoning like what you apply with products of two linear factors to write a rule for $q(x)$ in standard polynomial form. Record steps in your work so that someone else could check your reasoning.

c. Identify the degree of $q(x)$. How could you have predicted that property of the polynomial before any algebraic multiplication?

d. Graph $q(x)$ and label the x-intercepts, y-intercept, and local maximum and local minimum points with their coordinates.

3 The graph below is that of a cubic polynomial $c(x)$.



a. Use information from the graph to write a possible rule for $c(x)$. Express the rule in equivalent factored and standard polynomial forms.

b. Compare the overall shape of the graph, the local max/min points, and intercepts of the graph produced by your rule to the given graph. Adjust the rule if needed to give a better fit.

4 Look back at your work on Problems 1–3 to develop conjectures about answers to the following questions. In each case, be prepared to give other specific examples illustrating your idea.

a. How can you tell the zeroes of a polynomial function when its rule is written as a product of linear factors?

b. How can you tell the degree of a polynomial function when its rule is written as a product of linear factors?

c. Which properties of a polynomial function and its graph are shown best when the rule is written as a product of linear factors? When the rule is written in standard form?

"Advanced" Multiplication Early in your experience with variables and expressions, you learned that the Distributive Property of Multiplication over Addition guarantees correctness of statements like $3(x + 4) = 3x + 12$. More recently, you have used the distributive property to expand products of binomials like $(2x + 4)(x - 3) = (2x + 4)x - (2x + 4)3 = 2x^2 + 4x - 6x - 12 = 2x^2 - 2x - 12$.

5 Suppose that you were asked to expand the product $(2x + 1)(3x^2 + 2x + 4)$.

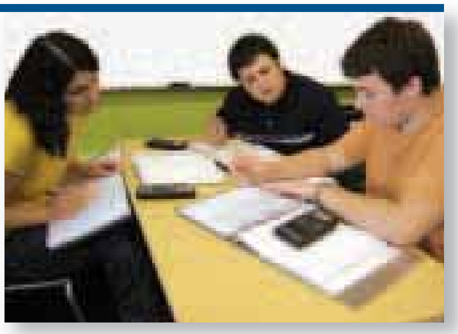
CPMP-Tools

a. How could you adapt the algorithm for finding the product of two binomials to find the desired product of polynomial factors? What standard polynomial form result does it give in this case?

b. Compare the zeroes of the factors $p(x) = 2x + 1$ and $q(x) = 3x^2 + 2x + 4$ and the product $p(x)q(x)$ in Part a.

6 When students in a *Core-Plus Mathematics* class in Denver, Colorado, were challenged to multiply a quadratic expression and a cubic expression, one group started like this.

$$(3x^2 + 4x - 10)(x^3 - 2x + 1) = 3x^5 - 6x^3 + 3x^2 + \cdots$$



a. Do you think they were on the right track? Explain your reasoning.

b. Based on what you know about the distributive, commutative, and associative properties, write the complete product in the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$. You might choose to find the result using a CAS and then figure out why it works that way, or use your own reasoning first and check that a CAS gives the same result.

c. What strategy would you recommend for keeping track of results in the process of multiplying two such nonlinear polynomials?

d. Identify the degree of the product polynomial. How could you figure this out before carrying out any multiplication?

e. The zeroes of $f(x) = 3x^2 + 4x - 10$ are approximately 1.28 and -2.61, and the zeroes of $g(x) = x^3 - 2x + 1$ are approximately 1, 0.62, and -1.62. Are those values of x also zeroes of the product function $(3x^2 + 4x - 10)(x^3 - 2x + 1)$? Check by substituting the possible zeroes in the polynomial produced by your work on Part b.

7 Write the following products in standard polynomial form. Identify the degree of each product. Explain how that degree is related to the degrees of the factors.

a. $(x - 4)(x^4 - 3x^2 + 2)$

b. $(x^5 + 7)(-2x^2 + 6x - 1)$

c. $(x^6 - 5x^5 + 3x^4 + 7x^3 - 6x^2 + 2x - 8)(x^2 + 7x + 12)$

Repeated Zeroes You may have noticed that in Problems 1–7, each first-degree polynomial function had one zero, each second-degree polynomial function had two zeroes, and each third-degree polynomial function had three zeroes. As you discovered earlier in the case of quadratic functions, this correspondence of polynomial degree and function zeroes is not always the case.

8 Consider the functions $r(x)$ and $s(x)$, where $r(x) = (x - 3)^2$ and $s(x) = (x + 3)^3$.

a. Expand the expressions that define $r(x)$ and $s(x)$. Identify the degree of each polynomial.

b. How many zeroes do $r(x)$ and $s(x)$ each have?

c. How many zeroes does the product $p(x) = r(x)s(x)$ have?

d. Use sketches of the respective function graphs to illustrate your responses in Parts b and c.

9 Consider the function $t(x) = (x - 3)(x + 4)^2$.

a. Expand the expression that defines $t(x)$. Identify the degree of the resulting polynomial.

b. What are the zeroes of $t(x)$?

c. Sketch a graph of $t(x)$ to show that the number of zeroes and the degree of the polynomial are not the same. Explain why that happens.

Summarize the Mathematics

In this investigation, you discovered connections between factors, zeroes, and the expanded form of polynomial functions.

a How is the degree of a product of polynomials determined by the degrees of its factors?

b What strategies can be used to find the standard form of a product of polynomials?

c How are the zeroes of a polynomial function related to the zeroes of its factors?

d In what cases will the degree of a polynomial function not equal the number of its zeroes?

Be prepared to explain your ideas to the class.

Check Your Understanding

Use your understanding of polynomial multiplication to complete these tasks.

a. Rewrite the following products in standard form
$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$.
Identify the degree of each result. Explain how that degree is related to the degrees of the factors.

i. $f(x) = (x^2 - 5x + 6)(x - 4)$

ii. $g(x) = (-2x^2 + 6x - 1)(x^5 + 7)$

b. What are the zeroes of the function $h(x) = (x^2 - 5x + 6)(2x - 7)$?

c. What are the zeroes of the function $j(x) = (x + 3)^2(2x - 5)$?

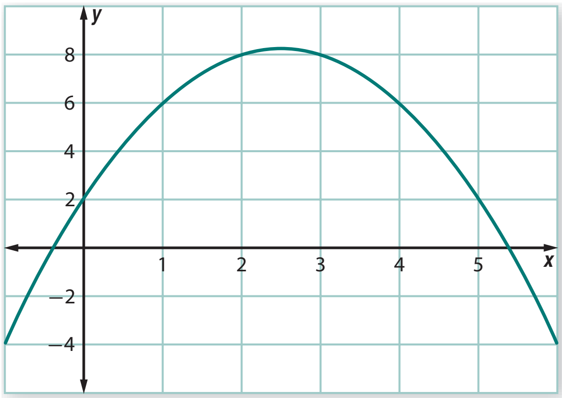
d. Write rules for quadratic and cubic polynomial functions that each have zeroes at $x = 5$ and $x = -2$ only.

On Your Own

Applications

1 In Parts a–d, use a curve-fitting tool and/or algebraic reasoning to find rules for functions $f(x)$, $g(x)$, $h(x)$, and $j(x)$ that model the given graph patterns. In each case, report the graph points used as the basis of your curve-fitting, the rule of the modeling function, and your reasons for choosing a model of that type.

a.



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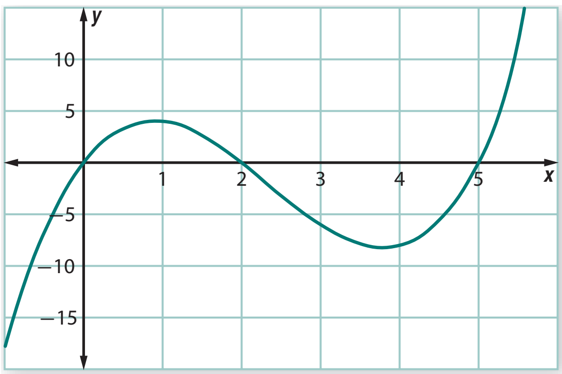

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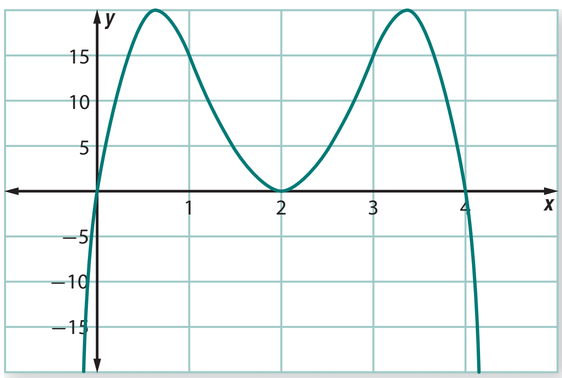

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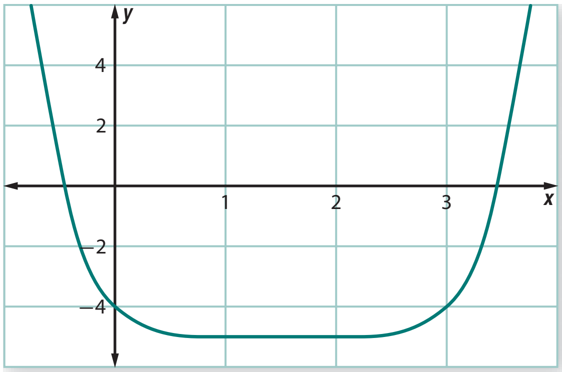


2 Graph each function and then calculate or estimate coordinates of all:

- local maximum points.
- local minimum points.
- x-intercepts.
- y-intercepts.

a. $f(x) = 2x^2 + 4x + 1$

b. $g(x) = x^3 + 2x^2 + 3x + 7$

c. $h(x) = x^3 - 6x^2 + 12x - 8$

d. $s(x) = x^4 - 8x^3 + 20x^2 - 16x$

3 For each algebraic expression:

- write an equivalent expression in standard polynomial form.
- identify the degrees of the expressions being combined and the degree of the result.

a. $(2x^2 + 5x - 2) + (-2x^2 + 3x + 7)$

b. $(2x^2 + 5x - 2) - (5x + 7)$

c. $(-7x^3 + 6x^2 + 3x - 7) - (3x^4 + 7x^3 + 4x^2 - 3x + 2)$

d. $(2x^2 + 3x - 7 + 5x^5 - 3x^4 - 7x^3) + (3x^5 + 2x^4 + 2x^3 + 4x^2 + 6x + 1)$

e. $(5x^5 - 3x^4 + 2x^2 + 6x - 7) + (7x^3 - 2x^2 - 3x + 5)$

4 Without using a graph, table, or CAS, find all zeroes of these functions.

a. $f(x) = (x - 3)(x + 4)$

b. $g(x) = (x^2 - 9)x$

c. $h(x) = (x^2 + 5)x^2$

d. $s(t) = (t + 5)^2$

5 For each algebraic expression:

- write an equivalent expression in standard polynomial form.
- identify the degrees of the expressions being combined and the degree of the result.

a. $(7x + 3)(x - 1)$

b. $(3x + 5)^2x^2$

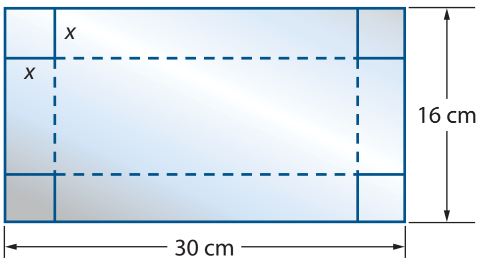
c. $(2x^2 + 3x - 7)(3x + 7)$

d. $(7x^3 - 6x + 4)(2x^2 - 7)$

e. $(7x^3 + 2)(5x^2 + 3x - 8)$

f. $(-3x^4 + 2x^2 + 6x)(7x^3 - 2x^2 + 5)$

6 *Great Lakes Supply* has been contracted to manufacture open-top rectangular storage bins for small electronics parts. The company has been supplied with 30 cm × 16 cm sheets of material. The bins are made by cutting squares of the same size from each corner of a sheet, bending up the sides, and sealing the corners.



a. On centimeter graph paper, draw a rectangle 30 cm long and 16 cm wide. Cut equal-size squares from the corners of the rectangle. Fold and tape the sides as suggested above. Find the volume of your bin.

b. Write a rule expressing the volume V of a bin as a function of the length of the corner cutout x.

i. What is the degree of the polynomial function?

ii. What is a practical domain for the function?

iii. Why must the graph of $V(x)$ have a local maximum point?

c. If the company is to manufacture bins with the largest possible volume, what should be the dimensions of the bins? What is the maximum volume?

7 The *Galaxy Sport and Outdoor Gear* company has a climbing wall in the middle of its store. Before the store opened for business, the owners did some market research and concluded that the daily number of climbing wall customers would be related to the price per climb x by the linear function $n(x) = 100 - 4x$.

a. According to this function, how many daily climbing wall customers will there be if the price per climb is \$10? What if the price per climb is \$15? What if the climb is offered to customers at no cost?

b. What do the numbers 100 and −4 in the rule for $n(x)$ tell about the relationship between climb price and number of customers?

c. What is a reasonable domain for $n(x)$ in this situation? That is, what values of x are plausible inputs for the function?

d. What is the range of $n(x)$ for the domain you specified in Part c? That is, what are the possible values of $n(x)$ corresponding to plausible inputs for the function?

e. If the function $I(x)$ tells how daily income from the climbing wall depends on price per climb, why is $I(x) = 100x - 4x^2$ a suitable rule for that function?

8 The function $e(x) = 2x + 150$ shows how daily operating expenses for the *Galaxy Sport* climbing wall (Task 7) depend on the price per climb x.

a. Write two algebraic rules for the function $P(x)$ that gives daily profit from the climbing wall as a function of price per climb, (1) one that shows how income and operating expense functions are used in the calculation of profit and (2) another that is in simpler equivalent form.

b. Find $P(5)$. Explain what this result tells about climbing wall profit prospects.

c. What is a reasonable domain for $P(x)$ in this problem situation?

d. What is the range of $P(x)$ for the domain you specified in Part c?

e. Write and solve an inequality that will find the climb price(s) for which *Galaxy Sport and Outdoor Gear* will not lose money on operation of the climbing wall.

f. Find the price(s) that will yield maximum daily profit from the climbing wall.

Connections

9 Recall that graphs of quadratic polynomial functions of the form $f(x) = ax^2 + bx + c$ are called *parabolas*.

a. A basic geometric fact is that "two points determine a line." In general, how many points are needed to determine a parabola? Explain your reasoning.

b. Under what conditions, if any, will two points determine a parabola?

10 Evaluate each of these polynomial functions when $x = 10$.

$f(x) = 2x^2 + 3x + 7$

$g(x) = 5x^3 + x^2 + 5x + 4$

$h(x) = 3x^4 + 7x^3 + 9x^2 + 2x + 1$

$j(x) = 5x^5 + 3x^4 + 6x^3 + 2x^2 + 3x + 6$

a. Study the set of results. Explain why things turn out as they do in every case.

b. How does the pattern observed in Part a help to explain the arithmetic guideline that when adding numbers like 2,351.7 and 462.23, you should always "line up the decimal points"?

11 The following work illustrates one method for multiplying whole numbers.

$$\begin{array}{r} 5{,}283 \\ \underline{\times\ 25} \\ 15 \\ 400 \\ 1{,}000 \\ 25{,}000 \\ 60 \\ 1{,}600 \\ 4{,}000 \\ \underline{100{,}000} \\ 132{,}075 \end{array}$$

a. What products are represented by the numbers 15, 400, 1,000, … , 100,000?

b. Use a similar procedure to calculate 1,789 × 64, recording all of the individual products that must be summed to get the final result.

c. Why does the procedure give a correct result?

d. How does this method for arithmetic multiplication relate to the procedure for multiplying two polynomials?

12 In this task, you will explore related methods for multiplication of numbers and polynomials.

a. The following table shows a method of organizing work in multiplication of numbers like 314×27.

	3	1	4
2	6	2	8
7	21	7	28

i. Why is the result of this operation equal to $6{,}000 + 2{,}300 + 150 + 28$?

ii. How are those partial products found from the table entries?

b. The next table shows a way of organizing the work involved in multiplying polynomials like $(7x^3 + 9x^2 + 2x + 1)$ and $(4x^2 + 3x + 5)$.

	7	9	2	1
4	28	36	8	4
3	21	27	6	3
5	35	45	10	5

i. What are the coefficients of each term in the result $\_\_x^5 + \_\_x^4 + \_\_x^3 + \_\_x^2 + \_\_x + \_\_$?

ii. How are the cell entries calculated?

iii. How are those cell entries combined to produce the coefficients of each term in the result? Why does that procedure give the desired result in each case?

13 Use the fact that $x^3 + 2x^2 - 11x - 12 = (x - 3)(x + 1)(x + 4)$ to solve the following inequalities. Express your solutions with inequality notation, number line graphs, and interval notation.

a. $x^3 + 2x^2 - 11x - 12 \geq 0$

b. $x^3 + 2x^2 - 11x - 12 < 0$

Reflections

14 One of the attractive features of polynomial models for data or graph patterns is the fact that evaluation of a polynomial requires only repeated use of three basic operations of arithmetic—addition, subtraction, and multiplication.

a. How does that make polynomials different and easier to use than some other types of functions that you have studied?

b. How do calculators and computers make that special feature of polynomials less important than it was before such technology was available?

15 Shown below are four polynomial functions, each written in expanded standard form and as a product of linear factors.

$$f(x) = x^4 - 10x^3 + 35x^2 - 50x + 24 = (x - 1)(x - 2)(x - 3)(x - 4)$$
$$g(x) = x^4 - 7x^3 + 17x^2 - 17x + 6 = (x - 1)^2(x - 2)(x - 3)$$
$$h(x) = x^4 - 5x^3 + 9x^2 - 7x + 2 = (x - 1)^3(x - 2)$$
$$j(x) = x^4 - 4x^3 + 6x^2 - 4x + 1 = (x - 1)^4$$

a. In which form is it easier to see the zeroes of the function?

b. In which form is it easier to see the y-intercept of the graph of the polynomial function?

c. How many zeroes do each of the four quartic polynomial functions have?

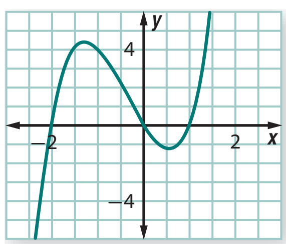
d. How does the collection of four functions help to explain why any polynomial function of degree n can have at most n distinct zeroes?

16 Graphs of quadratic polynomial functions have one basic shape—a parabola. For higher-degree polynomials, graphs of two polynomial functions of the same degree can have different shapes.

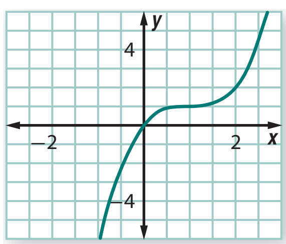
a. How does the basic shape of a quadratic polynomial function explain why quadratic functions can have 0, 1, or 2 zeroes?

b. Find a *cubic* (degree 3) polynomial function whose graph matches that shown in each case.

i.



ii.



c. As each curve is shifted vertically, what happens to the number of zeroes of the corresponding function?

d. What are the possible numbers of zeroes for any cubic polynomial function? Explain your reasoning.

Extensions

17 You can work in many different ways to evaluate a polynomial function like

$$p(x) = 5x^5 + 3x^4 + 6x^3 + 2x^2 + 3x + 6$$

for some specific value of x. But to write a computer algorithm for that task, it is desirable to accomplish the work with a minimum number of instructions.

a. The following algorithm gives directions for evaluation of any polynomial. Follow the algorithm to evaluate $p(10)$. Use your calculator for the arithmetic.

Step 1. Enter the coefficient of the highest degree term.

Step 2. Multiply the result by x.

Step 3. Add the coefficient of the next lower degree term. (It might be zero.)

Step 4. If all coefficients have been used, go to Step 5. Otherwise, return to Step 2.

Step 5. Add the constant term and report the result.

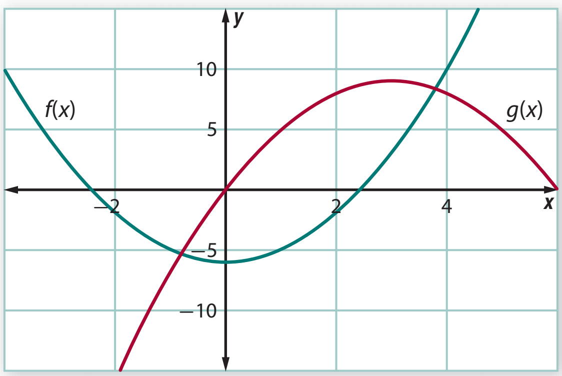
b. Write this expression in equivalent standard polynomial form.

$$((((5x + 3)x + 6)x + 2)x + 3)x + 6$$

c. Use the algorithm in Part a to evaluate $f(x) = 4x^4 + 5x^3 + 7x^2 + 1x + 9$ when $x = 3$.

d. Write an expression like that in Part b that expresses the calculations involved in evaluating the polynomial $4x^4 + 5x^3 + 7x^2 + 1x + 9$ for any specific x. Then show why your alternative form is equivalent to the original.

18 A polynomial function $f(x)$ can have a *global maximum* or a *global minimum* value. A **global maximum** is a number M with the property that $f(x) \leq M$ for all values of x. A **global minimum** is a number m with the property that $f(x) \geq m$ for all values of x. For example, the quadratic function $f(x) = x^2 - 6$ has a global minimum at the point $(0, -6)$. The function $g(x) = -x^2 + 6x$ has a global maximum at $(3, 9)$.



Use your experience with linear and quadratic functions and your explorations of graphs for cubic and quartic polynomials to answer these questions about global maximum and global minimum point possibilities. Be prepared to explain why you believe that your answers are correct.

a. Which polynomials of degree one, $p(x) = a_1x + a_0$, have global maximum or global minimum points?

b. Which polynomials of degree two, $p(x) = a_2x^2 + a_1x + a_0$, have global maximum or global minimum points?

c. Does the cubic polynomial function $g(x) = -x^3 + 6x^2 - 9x$ have a global maximum or global minimum?

d. Which polynomials of degree three, $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, have global maximum or global minimum points?

e. Does the quartic polynomial function $h(x) = -x^4 + 2x^3 + 7x^2 - 8x - 12$ have a global maximum or global minimum?

f. Which polynomials of degree four, $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, have global maximum or global minimum points?

19 When you first learned about division of whole numbers, you related that operation to multiplication by statements like "369 ÷ 3 = 123 because 3 × 123 = 369." You recorded the work to find a quotient in schemes that might have looked like this work.

$$\begin{array}{r} 123 \\ 3\overline{)369} \\ \underline{300} \\ 69 \\ \underline{60} \\ 9 \\ \underline{9} \end{array}$$

Apply your understanding and skill in division of whole numbers to find the quotients of polynomials in Parts a–d. It might help to organize your work in a style similar to what you learned to do with division of numbers, as shown in this example.

$$\begin{array}{r} x + 5 \\ x + 2\overline{)x^2 + 7x + 10} \\ \underline{x^2 + 2x}\phantom{{}+10} \\ 5x + 10 \\ \underline{5x + 10} \end{array}$$

a. $(x^2 + 11x + 28) \div (x + 7)$

b. $(x^2 + x - 12) \div (x - 3)$

c. $(x^3 + 7x^2 + 13x + 4) \div (x + 4)$

d. $(2x^3 + 13x^2 + 15x) \div (x^2 + 5x)$

20 In Investigation 3, you saw that if $(x - 3)$ is a factor of a polynomial function $q(x)$, then $q(3) = 0$.

a. Explain each step in the following proof of this general statement about polynomial functions.

If $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$.

(1) If $(x - a)$ is a factor of $f(x)$, then $f(x) = (x - a)g(x)$, where $g(x)$ is a polynomial.

(2) If $f(x) = (x - a)g(x)$, then $f(a) = 0$.

b. Write the converse of the if-then statement in Part a.

c. Combine the proposition proven in Part a and its converse (which can also be proven) into a single if-and-only-if statement. That statement is called the **Factor Theorem**.

Review

21 Rewrite each expression in simplest form.

a. $6x^2 - 10x + 5x^2 + x + 8$

b. $2x + 5 - (8x - 1)$

c. $3(x + 1) + 2(x^2 + 7) - 7x$

d. $(-6x^2)(9x^3)$

22 Solve each quadratic equation or inequality without the use of technology.

a. $x^2 - 4x + 4 = 0$

b. $x^2 + 6x + 8 < 0$

c. $x^2 - 4x - 5 \geq 0$

d. $x^2 + 14x + 42 = -7$

23 Laboratory tests indicate that when planted properly, 6% of a particular type of seed fail to germinate. This means that out of every 100 seeds planted according to instructions, on the average six do not sprout. The laboratory has been developing a new variety of the seed in which only 1% fail to germinate. Suppose that in an experiment, ten seeds of each of the two types are planted properly.

a. Calculate the theoretical probability that at least one seed out of the ten will fail to germinate for each variety of seed.

b. Design and carry out a simulation to estimate the chance that if ten of the seeds with the 6% germination rate are planted, at least one will fail to germinate.

c. Design and carry out a simulation to estimate the chance that if ten of the new variety of the seed are planted, at least one will fail to germinate.

d. Compare the estimates from your simulations to your calculations in Part a.

24 Write rules for quadratic functions whose graphs meet these conditions.

a. Opens up and has two x-intercepts

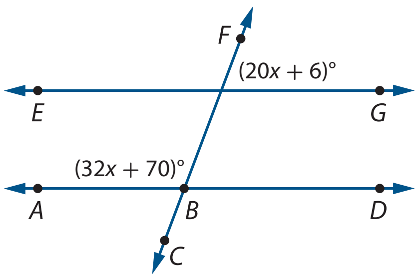
b. Opens down and has y-intercept $(0, -4)$

c. Opens up and the only x-intercept is $(5, 0)$

d. Has x-intercepts $(-2, 0)$ and $(6, 0)$

e. Opens up and has line of symmetry $x = 3$

25 In the diagram below, $\overleftrightarrow{AD} \parallel \overleftrightarrow{EG}$. Find m$\angle CBD$.



26 Rewrite each expression in equivalent simpler form with the smallest possible number under the radical sign.

a. $\sqrt{24}$

b. $\sqrt{48}$

c. $\sqrt{450}$

d. $\sqrt{\frac{8}{9}}$

e. $\frac{\sqrt{12}}{2}$

f. $\frac{\sqrt{60 + 3(5)}}{6}$

27 Write rules for linear functions whose graphs meet these conditions.

a. Slope 0.5 and y-intercept $(0, 4)$

b. Slope 1.6 and passing through $(2, 5)$

c. Passing through $(-2, 4)$ and $(6, 0)$

d. Parallel to the graph of $y = 2x + 3$ and passing through $(0, -3)$

28 Solve each of these quadratic equations by algebraic reasoning.

a. $x^2 + 7 = 23$

b. $3x^2 - 12 = 135$

c. $x^2 + 9 = 0$

d. $(x + 3)^2 = 19$

29 Expand each of these expressions to an equivalent standard form quadratic expression.

a. $(x + 5)(4x - 3)$

b. $(t + 8)(4 - t)$

c. $(x + 3)(x - 3)$

d. $(s - 7)^2$

e. $(3s + 2)^2$

f. $\left(y - \frac{5}{2}\right)^2$

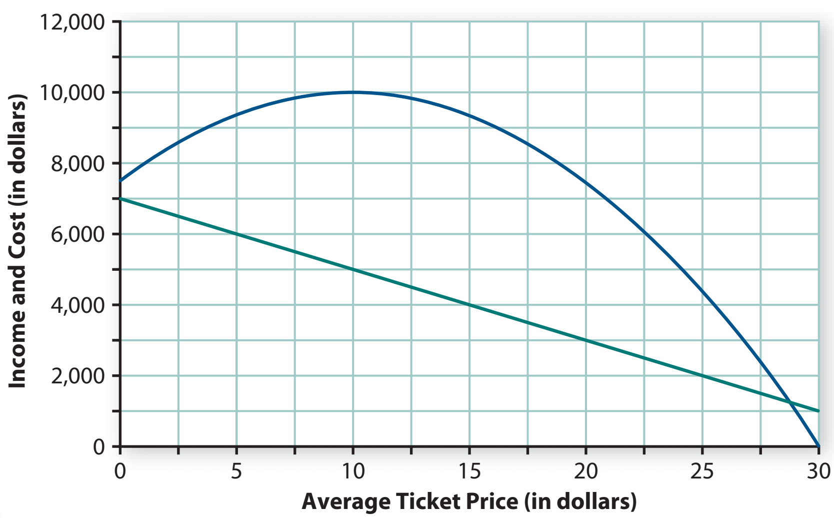
LESSON 2

Quadratic Polynomials

In the family of polynomial functions, the most useful are the simplest. In earlier work, you have used linear and quadratic polynomial functions to model and reason about a variety of business and scientific problems.

For example, in Lesson 1 of this unit, you analyzed business prospects for a concert promotion in which income was a quadratic function and operating expense was a linear function. Both depended on the price of tickets for the concert. The relationship of the two functions is shown below in the graph.

Concert Income and Operating Costs



For concert promoters, one of the critical planning questions is estimating the break-even point(s) at which income from ticket sales and operating expenses are equal. In this case, that involves solving the equation $-25x^2 + 500x + 7{,}500 = 7{,}000 - 200x$.

Think About This Situation

Think about the various algebraic reasoning and calculator- or computer-based methods that you have learned for work with linear and quadratic expressions and functions.

a What strategies can you apply to the tasks of finding linear or quadratic function rules that model conditions or data patterns in problems?

b What strategies can you apply to the tasks of solving linear or quadratic equations and inequalities?

c What strategies can you apply to find the maximum or minimum value of a quadratic function?

The investigations of this lesson will add to your toolkit of strategies for analyzing quadratic polynomials—writing quadratic expressions in new kinds of equivalent forms and using those expressions to solve equations and find max/min points on graphs of the corresponding functions.

Investigation 1 Completing the Square

When problems require solving quadratic equations, the simplest cases are those like $ax^2 + b = c$ that involve no linear term. You can solve such equations easily by reasoning like this.

$$\text{If } ax^2 + b = c, \text{ then } x^2 = \frac{c-b}{a}.$$

$$\text{So, } x = \pm\sqrt{\frac{c-b}{a}}.$$

It turns out that there are ways to transform every quadratic polynomial into an equivalent expression that has the general form $a(x - h)^2 + k$. This quadratic form, called the **vertex form**, gives easy-to-read information about the shape and location of the graph of the corresponding quadratic function. It also makes solution of quadratic equations easy.

As you work on the problems in this investigation, look for answers to these questions:

How does the vertex form of a quadratic function reveal the shape and location of its graph?

How can quadratic polynomials be expressed in vertex form?

As you work on these questions, keep in mind the fact that the general form $a(x - h)^2 + k$ includes examples like $-3(x + 6)^2 - 4$ where $a = -3$, $h = -6$, and $k = -4$.

Vertex Form of Quadratics To make use of quadratic polynomials in vertex form, you need to know how the parameters in those expressions are related to the shape and location of the corresponding function graphs.

1 First consider the question of how to find x- and y-intercepts on the graphs of quadratic functions with rules expressed in vertex form.

$$f(x) = a(x - h)^2 + k$$

To develop an answer to this question, you might analyze several specific examples like I–IV below and then make a conjecture. Then use algebraic reasoning to justify your conjecture. Or you might use more general algebraic reasoning to develop a conjecture and then test your ideas with the specific examples.

I. $g(x) = (x - 3)^2 - 16$ **II.** $j(x) = 3(x - 7)^2 - 12$

III. $m(x) = (x - 1)^2 + 25$ **IV.** $n(x) = -2(x + 6)^2 + 20$

a. What formula shows how to use values of a, h, and k to find coordinates for the x-intercepts on the graph of any function with rule in the form $f(x) = a(x - h)^2 + k$?

b. What formula shows how to use values of a, h, and k to find coordinates for the y-intercept on the graph of any function with rule in the form $f(x) = a(x - h)^2 + k$?

2 Consider next the question of how to locate maximum or minimum points on the graph of a quadratic function $f(x) = a(x - h)^2 + k$. Recall that this point is called the **vertex** of the graph. To develop an answer to this question, you might analyze several specific examples like I–IV below to make a conjecture. Then use algebraic reasoning to justify your conjecture. Or you might use more general algebraic reasoning to develop a conjecture and then test your ideas with the specific examples.

I. $r(x) = (x - 3)^2 + 5$ **II.** $s(x) = 3(x - 7)^2 - 4$

III. $t(x) = -2(x + 6)^2 + 3$ **IV.** $v(x) = -0.5(x + 5)^2 - 7$

a. How can you tell from the parameters a, h, and k when the graph of the function $f(x) = a(x - h)^2 + k$ will have a minimum point? When it will have a maximum point?

b. Consider the cases when the graph of the function has a minimum point.

i. For what value of x does the expression $a(x - h)^2 + k$ take on its smallest value?

ii. What is the y-coordinate of the corresponding point on the graph?

iii. What are the coordinates of the minimum point when $a > 0$?

c. Now consider the cases when the function has a maximum point.

i. For what value of x does the expression $a(x - h)^2 + k$ take on its largest possible value?

ii. What is the y-coordinate of the corresponding point on the graph?

iii. What are the coordinates of the minimum point when $a < 0$?

3 Your work on Problems 1 and 2 revealed a new way of thinking about max/min points and intercepts on graphs of quadratic functions. For example, the function $q(x) = x^2 - 10x + 24$ can be expressed as $q(x) = (x - 4)(x - 6)$ and also as $q(x) = (x - 5)^2 - 1$.

a. How would you use each of the three forms to find coordinates of the y-intercept on the graph of $q(x)$? Which approach seems most helpful?

b. How would you use each form to find coordinates of the x-intercepts on the graph of $q(x)$? Which approach seems most helpful?

c. How would you use each form to find coordinates of the maximum or minimum point on the graph of $q(x)$? Which approach seems most helpful?

Completing the Square You know from earlier work with quadratic expressions that there are useful patterns relating factored and expanded forms of those expressions. For any number n,

$$(x + n)^2 = x^2 + 2nx + n^2 \text{ and } (x - n)^2 = x^2 - 2nx + n^2.$$

The relationship between factored and expanded forms of such *perfect square* expressions is the key to finding vertex forms for quadratic function rules. Work on Problems 4–7 will give you some valuable insight into the process of finding vertex form expressions for quadratics.

4 Use the pattern relating expanded and factored forms of perfect square polynomials (where possible) to write factored forms of the expressions in Parts a–f. Use a CAS to check that your factored forms are equivalent to the corresponding expanded forms.

a. $x^2 + 6x + 9$

b. $x^2 + 10x + 25$

c. $x^2 + 8x + 12$

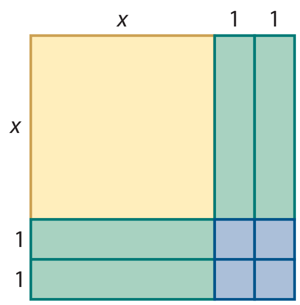
d. $x^2 - 10x + 25$

e. $x^2 + 5x + \frac{25}{4}$

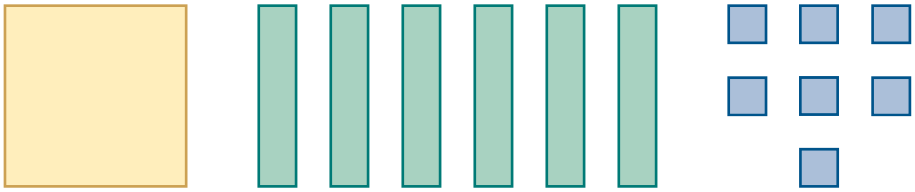
f. $x^2 - 3x + \frac{9}{4}$

5 The challenge in transforming a given quadratic expression to an equivalent vertex form can be represented as a geometry problem. Study the following diagram. Find values of n, m, and k that make it possible to express the total area of the figure in two ways.

- as $(x + n)^2$
- as $x^2 + kx + m$



6 Now consider the expression $x^2 + 6x + 7$. The pieces of this expression can be represented with an "x by x" square, six "x by 1" rectangles, and seven unit squares.



$x^2 + 6x + 7$

a. Why is it impossible to arrange the given squares and rectangles without overlapping to form one large square?

b. How could you produce the requested square if you were given some extra pieces or allowed to take away some pieces? What addition or subtraction would do the job most efficiently?

c. Explain how your answer to Part b illustrates the fact that $x^2 + 6x + 7 = (x + 3)^2 - 2$.

If a quadratic expression is not a perfect square (factorable to the form $(x \pm h)^2$), it is still possible to write an equivalent expression in the form $(x \pm h)^2 \pm k$. Your work in Problem 6 gives a picture that suggests the kind of algebraic operations that are required. Problem 7 asks you to adapt that reasoning to develop a symbol manipulation procedure.

7 Study the following example of the technique called **completing the square**. Explain the choice of "9" and why it was added and subtracted in different parts of the expression.

$$\begin{aligned} x^2 + 6x + 11 &= (x^2 + 6x + \underline{\quad}) + 11 \\ &= (x^2 + 6x + 9) - 9 + 11 \\ &= (x + 3)^2 + 2 \end{aligned}$$

Use similar reasoning to write the expressions in Parts a–f in vertex form.

a. $x^2 + 6x + 5$

b. $x^2 - 6x + 10$

c. $x^2 + 8x + 19$

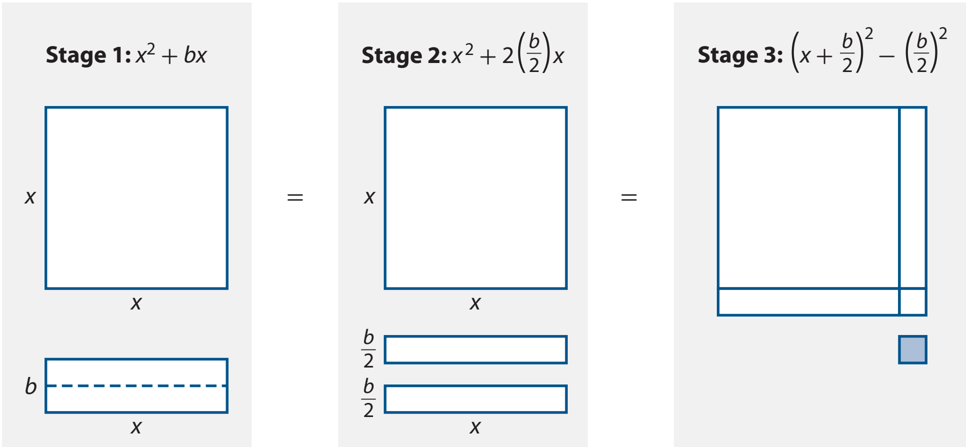
d. $x^2 - 8x + 12$

e. $x^2 + 10x + 27$

f. $x^2 - 3x + 1$

8 How do the results of your work on Problems 6 and 7 suggest a strategy for writing any quadratic expression $x^2 + bx + c$ in equivalent vertex form $(x \pm h)^2 \pm k$?

9 You can visualize the process of completing the square in a somewhat different way, if you focus attention on the first two terms of the given expression, $x^2 + bx$. Explain how the following proof without words shows that $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.



Summarize the Mathematics

In this investigation, you developed skill in transforming quadratic expressions in standard form to vertex form and in using that form to analyze the graphs of corresponding quadratic functions.

a How can the vertex form of a quadratic expression like $(x - h)^2 + k$ be used to locate the max/min point and intercepts on the graph of the corresponding quadratic function?

b What are key steps in transforming $x^2 + bx + c$ to vertex form?

Be prepared to explain your ideas to the class.

Check Your Understanding

Use your new completing-the-square skills to solve these problems.

a. Write the expression $x^2 - 2x - 5$ in equivalent vertex form.

b. Use the result of Part a to:

i. solve $x^2 - 2x - 5 = 0$.

ii. find coordinates of the minimum point on the graph of $f(x) = x^2 - 2x - 5$.

iii. find coordinates of x- and y-intercepts on the graph of $f(x) = x^2 - 2x - 5$.

Investigation 2 The Quadratic Formula and Complex Numbers

As you have seen in your previous studies, when problems require solving quadratic equations that cannot be factored easily, you can always turn to the *quadratic formula*. For instance, solutions for $3x^2 + 7x - 12 = 0$ are given by

$$x = \frac{-7}{2(3)} + \frac{\sqrt{7^2 - 4(3)(-12)}}{2(3)} \quad \text{and} \quad x = \frac{-7}{2(3)} - \frac{\sqrt{7^2 - 4(3)(-12)}}{2(3)}.$$

As you complete the problems in this investigation, look for answers to these questions:

How can the technique of completing the square be used to derive the quadratic formula?

How does use of the quadratic formula suggest the need for new kinds of numbers?

Proving and Using the Quadratic Formula The work that you have done to write quadratic expressions in vertex form is closely related to the quadratic formula that can be used to find solutions of any quadratic equation.

1. Consider the general form of a quadratic equation, $ax^2 + bx + c = 0$. The solutions of this equation are given by $x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$. Explain how each step in the following derivation of the quadratic formula is justified by properties of numbers and operations.

   Start: If $ax^2 + bx + c = 0$ (and $a \neq 0$),

   Step 1. Then $ax^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

   Step 2. $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

   Step 3. $x^2 + \frac{b}{a}x = \frac{-c}{a}$

   Step 4. $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$

   Step 5. $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} + \frac{-c}{a}$

   Step 6. $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-c}{a}$

   Step 7. $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-4ac}{4a^2}$

   Step 8. $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

   Step 9. $x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}$

   Step 10. So, $x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$.

2 The quadratic formula provides a tool for solving any quadratic equation by algebraic reasoning. But you have other helpful strategies available through use of technology.

a. How could you use calculator- or computer-generated tables of function values or graphs to estimate solutions for a quadratic equation like $3x^2 + 7x - 12 = 0$?

b. How could you use a computer algebra system (CAS) to solve $3x^2 + 7x - 12 = 0$?

c. Use the CAS available to you to solve $ax^2 + bx + c = 0$ for x and compare the CAS result to the quadratic formula derived in Problem 1.

CPMP-Tools

3 Use the quadratic formula to solve each of the following equations. Report your answers in exact form, using radicals where necessary rather than decimal approximations. Check each answer by substituting the solution values for x back into the original equation.

a. $3x^2 + 9x + 6 = 0$

b. $3x^2 - 6 = -3x$

c. $2x^2 + x - 10 = 0$

d. $2x^2 + 5x - 1 = 0$

e. $x^2 + 2x + 1 = 0$

f. $x^2 - 6x + 13 = 0$

4 The solutions for quadratic equations in Problem 3 included several kinds of numbers. Some could be expressed as integers. But others could only be expressed as fractions or as irrational numbers involving radicals. One of the quadratic equations appears to have no solutions.

a. At what point in use of the quadratic formula do you learn whether the equation has two distinct solutions, only one solution, or no real number solutions?

b. If the coefficients of a quadratic equation are integers or rational numbers, at what point in use of the quadratic formula do you learn whether the solution(s) will be integers, rational numbers, or irrational numbers?

Complex Numbers In work on Problems 2 and 3, you discovered that some quadratic equations do not have real number solutions. For example, when you try to solve $x^2 - 6x + 13 = 0$, the quadratic formula gives

$$x = 3 + \frac{\sqrt{-16}}{2} \quad \text{and} \quad x = 3 - \frac{\sqrt{-16}}{2}.$$

If you ask a CAS to **solve(x^2-6*x+13=0,x)**, it is likely to return the disappointing result "false."

5 Sketch a graph of the function $f(x) = x^2 - 6x + 13$. Explain how it shows that there are no real number solutions for the equation $x^2 - 6x + 13 = 0$.

The obstacle to solving $x^2 - 6x + 13 = 0$ appears with the radical $\sqrt{-16}$. Your prior experience with square roots tells you that *no* real number has -16 as its square. For thousands of years, mathematicians seemed to accept as a fact that equations like $x^2 - 6x + 13 = 0$ simply have no solutions. In general, a quadratic equation in the form $ax^2 + bx + c = 0$ has no real number solutions when the value of $b^2 - 4ac$, called the **discriminant** of the quadratic, is a negative number.

In the middle of the 16th century, Italian scholar Girolamo Cardano suggested all that was needed was a new kind of number. Cardano's idea was explored by mathematicians for several more centuries (often with strong doubts about using numbers with negative squares) until *complex numbers* became an accepted and well-understood tool for both pure and applied mathematics.


Girolamo Cardano

The obstacle to solving $x^2 - 6x + 13 = 0$ was removed by reasoning like this.

It makes sense that $\sqrt{-16}$ should equal $\sqrt{16(-1)}$.

But $\sqrt{16(-1)}$ should equal $\sqrt{16}\sqrt{-1}$.

So, $3 + \frac{\sqrt{-16}}{2}$ should equal $3 + \frac{4\sqrt{-1}}{2}$.

Or, the solutions for $x^2 - 6x + 13 = 0$ should be $x = 3 + 2\sqrt{-1}$ and $x = 3 - 2\sqrt{-1}$.

This kind of observation led mathematicians to develop what is now called the **complex number** system whose elements are in the form $\boldsymbol{a + b\sqrt{-1}}$ with a and b real numbers. The new number system contains all real numbers (the complex numbers for which $b = 0$). It also provides solutions to the problematic quadratic equations *and all other polynomial equations in the form $p(x) = 0$.*

Because $\sqrt{-1}$ was long considered an impossible or **imaginary number**, expressions of complex numbers are commonly written with $\sqrt{-1}$ replaced by the letter "i". So, $3 + 2\sqrt{-1}$ is written as $3 + 2i$. You will learn more about complex numbers in future studies.

6 Use the quadratic formula to show that each of these equations has complex number solutions with nonzero imaginary parts. Express those solutions in the form $a + bi$, where a and b are real numbers.

a. $x^2 - 10x + 29 = 0$

b. $4x^2 + 16 = 0$

c. $x^2 - 4x + 13 = 0$

d. $3x^2 - 18x + 30 = 0$

Summarize the Mathematics

In this investigation, you used the technique of completing the square to derive the quadratic formula and practiced use of that formula in solving quadratic equations.

a. What is the quadratic formula for solving equations in the form $ax^2 + bx + c = 0$?

b. What does the value of the discriminant $b^2 - 4ac$ tell you about the number and type of solutions to the quadratic equation $ax^2 + bx + c = 0$?

c. How will the graph of $f(x) = ax^2 + bx + c$ tell when the quadratic equation $ax^2 + bx + c = 0$ has 2, 1, or 0 real number solutions? What will the graph tell about the value of the discriminant $b^2 - 4ac$?

Be prepared to explain your ideas to the class.

Check Your Understanding

Use the quadratic formula or other reasoning to solve each of the following equations. If the solutions are real numbers, identify them as integer, noninteger rational, or irrational numbers. Write nonreal complex number solutions in *standard form* $a + bi$.

a. $x^2 - 6x - 7 = 0$

b. $5x^2 - 6x + 2 = 0$

c. $6x^2 - 11x - 10 = 0$

d. $5(x - 3)^2 + 6 = 11$

Applications

1 Find coordinates of the max/min points, x-intercepts, and y-intercepts on graphs of these functions.

a. $f(x) = (x + 2)^2 - 9$

b. $g(x) = -(x - 2)^2 + 3$

c. $h(x) = (x - 5)^2 - 2$

d. $j(n) = (n + 7)^2 + 4$

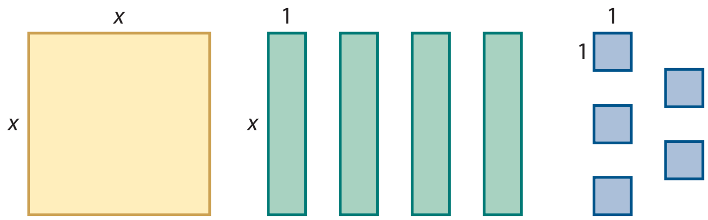
2 In a *Punkin' Chunkin'* contest, the height (in feet) of shots from one pumpkin cannon is given by the function $h(t) = -16(t - 5)^2 + 425$. The height is in feet above the ground and the time is in seconds after the pumpkin leaves the cannon. Show how to use this function to answer questions about the height of the flying pumpkin.

a. At what height is the pumpkin released from the "chunker"?

b. At what time will the pumpkin hit the ground?

c. At what time does the pumpkin reach its maximum height and what is that height?

3 Consider the collection of squares and rectangles shown here.



The four rectangles are congruent to each other, as are the five small squares.

a. What expression represents the total area of the ten figures?

b. Explain why the given shapes cannot be arranged to form a larger square without overlaps or gaps.

c. What is the minimal addition or subtraction of unit squares that will make it possible to arrange the new set of pieces to form a larger square?

d. Write an expression involving a binomial square to represent the area of the larger square formed by all the given pieces and the unit squares added or subtracted in Part c.

4 For each function, write the rule in vertex form and find coordinates of the max/min point on its graph.

a. $f(x) = x^2 + 12x + 11$

b. $g(x) = x^2 - 4x + 7$

c. $h(x) = x^2 - 18x + 74$

d. $j(s) = s^2 + 2s + 5$

5 Use the results of Task 4 to solve each of these equations.

a. $x^2 + 12x + 11 = 0$

b. $x^2 - 4x + 7 = 19$

c. $x^2 - 18x + 74 = 13$

d. $s^2 + 2s + 5 = 53$

6 Use the quadratic formula to solve each of these equations. If the solutions are real numbers, identify them as integer, noninteger rational, or irrational numbers. Write nonreal complex number solutions in standard form $a + bi$.

a. $2x^2 + 3x - 5 = 0$

b. $2x^2 + x - 3 = 0$

c. $3x^2 + x = 10$

d. $5x + x^2 + 10 = 0$

e. $x^2 + 9x - 10 = -24$

f. $3x^2 + 10 = 25$

7 Recall what you learned in Lesson 1 about factors of polynomials and zeroes of the corresponding polynomial functions. Use that knowledge to write quadratic equations in the form $ax^2 + bx + c = 0$ with two distinct solutions that:

a. are both integers.

b. include at least one fraction.

c. are both irrational numbers.

8 Use your knowledge of the quadratic formula to write quadratic equations with the indicated solutions.

a. exactly one real number solution

b. solutions that are complex numbers in the form $a + bi$, $a \neq 0$, $b \neq 0$

c. solutions that are imaginary numbers bi

9 The functions $I(x) = -25x^2 + 500x + 7{,}500$ and $E(x) = 7{,}000 - 200x$ give income and operating expenses as functions of average ticket price for a music concert. Use the quadratic formula to solve the equation $-25x^2 + 500x + 7{,}500 = 7{,}000 - 200x$ and find the ticket price(s) that will allow the promoters to break even on the event. That is, find the average ticket price that will make income equal to operating expenses.

Connections

10 There is a useful connection between the completing the square technique and coordinate equations for circles.

a. What are the center and radius for the circle with coordinate equation $(x - 5)^2 + (y - 7)^2 = 9$?

b. What are the center and radius for the circle with coordinate equation $(x + 5)^2 + (y + 7)^2 = 4$?

c. Write the equation in Part a without parentheses in the form $x^2 + y^2 + ax + by + c = 0$.

d. Write the equation in Part b without parentheses in the form $x^2 + y^2 + ax + by + c = 0$.

e. Use what you know about completing the square to write each of the following equations in a form that gives the center and radius of the circle. State the center and radius.

i. $x^2 + y^2 - 4x - 8y + 19 = 0$

ii. $x^2 + y^2 + 10x + 2y - 10 = 0$

iii. $x^2 + y^2 + 4y - 45 = 0$

iv. $x^2 + y^2 - 8x + 6y = 0$

11 Sketch graphs of these quadratic functions. Explain how the collection of graphs show that quadratic equations can have: (1) two real number solutions, (2) one repeated real number solution, or (3) no real number solutions.

a. $f(x) = x^2 - 4x - 5$

b. $g(x) = x^2 - 4x + 4$

c. $h(x) = x^2 - 4x + 6$

12 Solve each of these equations. Then compare the forms of the given equations and the procedures used to solve them.

a. $3m + 12 = 27$

b. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

c. $3n^2 + 12 = 27$

13 The complex number system was constructed in stages that began with the set **W** of whole numbers {0, 1, 2, 3, 4, … } and gradually introduced other important sets of numbers. Make a Venn diagram that illustrates the relationship among the following sets of numbers: whole numbers, integers, rational numbers, irrational numbers, real numbers, imaginary numbers, and complex numbers.

14 Each complex number can be represented in the form $a + bi$, where a and b are real numbers and i represents the imaginary $\sqrt{-1}$. At the turn of the 19th century, three different mathematicians, Caspar Wessel (1797), Jean Robert Argand (1806), and Carl Friedrich Gauss (1811), suggested that the complex numbers could be represented by points on a coordinate plane. The complex number $a + bi$ would correspond to the point with coordinates (a, b).

a. On a single coordinate diagram, locate and label points corresponding to these complex numbers.

$$4 + 2i \qquad -2 + 3i \qquad -4 - 3i \qquad 2 - 3i$$

b. Use of commutative and associative properties of addition and combining "like terms" suggests a rule for addition of complex numbers, $(a + bi) + (c + di) = (a + c) + (b + d)i$. Use this rule to add the complex number $3 + 2i$ to each of the numbers in Part a.

c. Using a second color, plot and label the points corresponding to your results in Part b. Then identify the geometric transformation that is accomplished by adding $3 + 2i$ to every complex number $a + bi$.

Reflections

15 When one student was asked to solve the quadratic equation $2x^2 + 12x = 5$, he wrote the factored form $2x(x + 6) = 5$ and concluded that the solutions are $x = 0$ and $x = -6$. What is the probable error in his thinking that led him to those incorrect answers?

16 You now know three different ways to express quadratic polynomials in equivalent forms—standard form, factored form, and vertex form.

a. What do you see as the advantages of each form?

b. Do you think that the value added by use of each different form is equal to the effort involved in transforming the original expression to a new form?

17 The creative team of *Conklin Brothers Circus* designed a cannon to shoot a human cannonball across the arenas where they operate. They judged that the height of the "cannonball" would be a quadratic function of time and wrote three different rules for the function (time in seconds and height in feet).

$$h(t) = -16t^2 + 32t + 48$$
$$h(t) = -16(t + 1)(t - 3)$$
$$h(t) = -16(t - 1)^2 + 64$$

a. In what different ways could you convince yourself or others that the three different rules are mathematically equivalent?

b. Which of the rules would be most helpful in answering each of these questions?

  i. What is the maximum height of the "cannonball" and when will it occur in the flight?

  ii. When would the "cannonball" hit the ground if the net collapsed while the flight was underway?

  iii. At what height does the "cannonball" leave the end of the cannon?

18 How do you decide on an approach to solving a quadratic equation? What conditions influence your strategy choice in various situations?

19 One solution of a quadratic equation $ax^2 + bx + c = 0$ is $2 + 3i$. What is the other solution? Explain your reasoning.

CPMP-Tools

Extensions

20 Of the two algebraic operations, expanding a product of linear factors is generally much easier than factoring a standard form quadratic. To understand factoring, you have to develop some skill in expanding products and in reversing that process.

Use algebraic reasoning to write each of these products in standard expanded form. Then check your results by using a CAS for the same work.

a. $(x + 4)(3x + 1)$ **b.** $(2x + 4)(x + 3)$

c. $(2x - 4)(5x + 3)$ **d.** $(x - 7)^2$

e. $(3x + 4)^2$ **f.** $(2x - 5)^2$

21 Use your experience in Task 20 to solve these quadratic equations by first writing the quadratic expression in equivalent form as a product of two linear factors.

a. $2x^2 + 7x + 3 = 0$ **b.** $5x^2 + 16x + 3 = 0$

c. $5x^2 + 17x = -6$ **d.** $6x^2 + 19x + 15 = 0$

e. $4x^2 - 12x + 9 = 0$ **f.** $9x^2 + 6x + 1 = 0$

22 Solve each of the following equations using factoring, where possible. Use a CAS to check your work.

a. $4x^2 + 12x + 9 = 0$ **b.** $9x^2 + 30x + 25 = 0$

c. $7x^2 - 18x + 11 = 0$ **d.** $9x^2 - 12x = -4$

e. $4x^2 + 10x + \frac{25}{4} = 0$ **f.** $4x^2 - 6x + \frac{9}{4} = 0$

23 When $a \neq 1$, quadratic functions in the form $f(x) = ax^2 + bx + c$ can still be transformed to equivalent vertex forms like $f(x) = m(x \pm n)^2 \pm d$.

a. Study the following example. Explain the choice of adding and subtracting 18 in two parts of the quadratic polynomial.

$$\begin{aligned} f(x) &= 2x^2 + 12x + 14 \\ &= 2(x^2 + 6x + \_\_) + 14 \\ &= 2(x^2 + 6x + 9) - 18 + 14 \\ &= 2(x + 3)^2 - 4 \end{aligned}$$

b. Use similar reasoning to write each function rule below in vertex form.

i. $f(x) = 3x^2 + 12x + 15$

ii. $g(x) = 5x^2 + 30x + 50$

iii. $h(x) = 2x^2 - 8x + 10$

iv. $j(x) = 4x^2 - 20x + 7$

c. How do the results of your work in Part b suggest a strategy for writing any quadratic function $f(x) = ax^2 + bx + c$ in equivalent vertex form $f(x) = m(x \pm n)^2 \pm d$?

24 Use the Factor Theorem (Extensions Task 20 in Lesson 1) to help explain why a cubic polynomial function has either zero or two nonreal complex number solutions.

25 Consider the ways that it would make sense to multiply complex numbers like $(3 + 2i)$ and $(4 + 7i)$.

a. If you think of the given complex numbers as linear expressions like $(3 + 2x)$ and $(4 + 7x)$, what would the product look like when expanded?

b. Because the letter i is the number for which $i^2 = -1$, the expression in Part a can be simplified. Write the product $(3 + 2i)(4 + 7i)$ in standard complex number form.

Review

26 Express each of these sums and differences as single fractions in simplest form.

a. $\frac{1}{3} + \frac{1}{6}$

b. $\frac{1}{2} - \frac{1}{8}$

c. $\frac{1}{4} + \frac{1}{3}$

d. $\frac{2}{3} - \frac{1}{2}$

e. $\frac{a}{b} + \frac{c}{d}$

27 Consider the following numbers.

$$\sqrt{100} \qquad \frac{12}{5} \qquad -5 \qquad 2.2 \qquad \sqrt{5}$$
$$\pi \qquad 3.\overline{1} \qquad \frac{1}{6} \qquad \sqrt{2.25} \qquad \sqrt[3]{27}$$

a. Which of the listed numbers are integers?

b. Which of the listed numbers are rational numbers?

c. Which of the listed numbers are irrational numbers?

d. Place the numbers in order from smallest to largest.

CPMP-Tools

28 Find equivalent expanded forms for these expressions. Then use a CAS to check your reasoning. (*Remember*: The algebraic product mx must generally be entered as **m*x** in a CAS.)

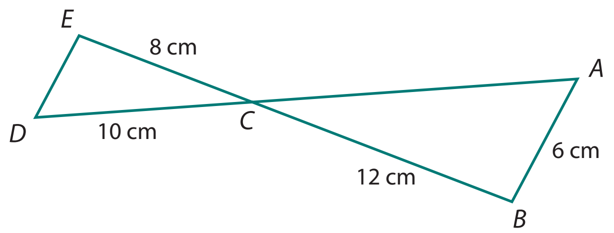
a. $(x + n)^2$

b. $(mx + n)^2$

c. $(x + n)(x - n)$

d. $(mx + n)(mx - n)$

29 In the diagram at the right, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$.



a. Prove that $\triangle ABC \sim \triangle DEC$.

b. Find the lengths of $\overline{AC}$ and $\overline{DE}$.

30 Without using a graphing calculator or a computer graphing tool, sketch a graph of each power function. Then check your ideas using technology.

$$f(x) = \frac{3}{x} \qquad\qquad g(x) = \frac{3}{x^2}$$

a. What is the domain of $f(x)$? Of $g(x)$?

b. What is the range of $f(x)$? Of $g(x)$?

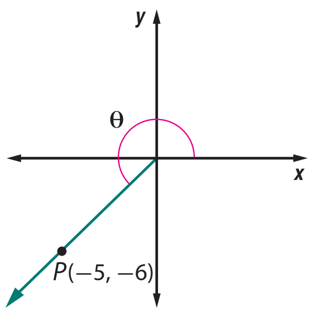
c. Describe the behavior of the graph of $f(x)$ as $|x|$ gets very large. As $|x|$ gets very small.

d. Describe the behavior of the graph of $g(x)$ as $|x|$ gets very large. As $|x|$ gets very small.

31 Simplify each algebraic expression.

a. $\dfrac{6x - 10}{2}$ **b.** $\dfrac{24 - 3x}{3}$ **c.** $\dfrac{x - 4}{x - 4}$

32 Consider the diagram below. Point P is on the terminal side of an angle θ in standard position. Find $\sin \theta$, $\cos \theta$, and $\tan \theta$.



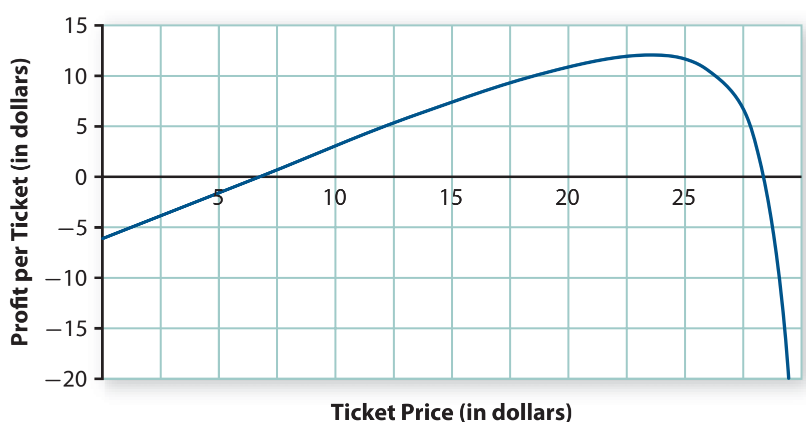
LESSON 3

Rational Expressions and Functions

How valuable is each customer to the bottom line of a business? In Lesson 1 of this unit, you discovered how income, expenses, and profits for a concert were functions of concert ticket price. It is possible to combine those functions to see how the profit from *each* ticket is related to the ticket price, giving a measure of each customer's value to the business.

The profit-per-ticket idea might seem simple, but the resulting function has some surprising properties. For example, if you focus only on income and expenses related to the concert itself (ignoring snack bar operations), the profit-per-ticket function has a graph that looks like the one below.

Profit per Ticket from Ticket Sales



Think About This Situation

Examine the profit-per-ticket graph on the previous page.

a How would you describe the pattern of change in profit per ticket as ticket prices approach $30?

b When the ticket price is set at $15, the business plan predicts that 375 tickets will be sold. Total profit from ticket sales is predicted to be $2,750. How would you calculate the profit per ticket sold when tickets are $15? How is your answer shown in the graph?

c How would you find a rule for calculating profit per ticket at any ticket price?

The strange properties of the profit-per-ticket function result from the fact that it is a *rational function*, the quotient of two polynomials. In this lesson, you will explore the family of rational functions and learn how to use them as mathematical models of problem situations. You will learn the connections between rules and graphs for rational functions and how to simplify and combine the expressions in those rules.

Investigation 1 Domains and Graphs of Rational Functions

Information from analysis of the concert business led to the prediction that ticket sales and profit from concert operation alone (not including snack bar operations) would be related to average ticket price x (in dollars) by these rules.

Number of tickets sold: $n(x) = 750 - 25x$
Profit from ticket sales: $P_c(x) = -25x^2 + 875x - 4{,}750$

As you complete the following problems, look for answers to these questions:

How can polynomials like these be combined to give useful rational functions?

What are the important features of rational functions and their graphs?

1 Suppose that the concert promoters want to estimate the profit per ticket for various possible ticket prices.

a. What number of tickets, total profit, and profit per ticket would be expected:

i. if the average ticket price x is $10?

ii. if the average ticket price x is $20?

b. Write an algebraic rule for the function $R(x)$ giving profit per ticket when the ticket price is x dollars. Express your answer as an algebraic fraction.

A function with rule that is a fraction in which both numerator and denominator are polynomials is called a **rational function**. That is, a function $f(x)$ is a rational function if and only if its rule can be written in the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions.

The profit-per-ticket function $R(x)$ that you developed in Problem 1 is a rational function. Like many other rational functions, its graph has several interesting and important features. To understand those properties, it helps to begin with analysis of the functions that are the numerator and the denominator of $R(x)$.

2 Sketch graphs of $n(x) = 750 - 25x$ and $P_c(x) = -25x^2 + 875x - 4{,}750$. Use the graphs to answer the following questions.

a. What is the practical domain of $n(x)$? That is, what prices x will give predicted numbers of tickets that make sense in the concert situation?

b. Estimate the ticket price for which profit from concert operation $P_c(x)$, excluding snack bar operations, is maximized. Find that profit.

3 Now turn to analysis of the profit-per-ticket function $R(x)$.

a. Graph $R(x)$ for x between 0 and 30.

b. Estimate the ticket price for which profit per ticket is maximized.

c. Why is the profit per ticket not maximized at the same ticket price that maximizes concert profit?

4 Something unusual happens in the graph of $R(x)$ near $x = 30$. Use a calculator or computer software to examine values of $R(x)$ when x is very close to 30, say, between 29 and 31 with increments of 0.1. If using a calculator, use **Dot** rather than **Connected** mode.

a. How does your technology tool respond to requests for $R(30)$? What does the response mean? Why does it make sense in this situation?

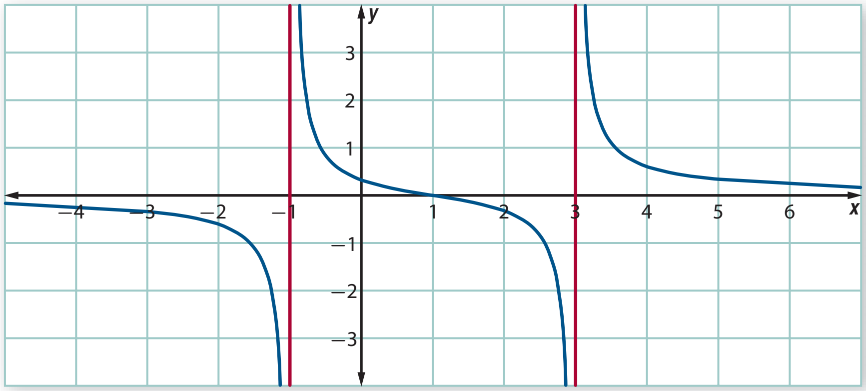
b. Describe the pattern of change in values of $R(x)$ as x approaches very close to 30 from below. Explain the trend by referring to the values of $P_c(x)$ and $n(x)$ that are used to calculate $R(x)$.

c. Describe the pattern of change in values for $R(x)$ as x approaches very close to 30 from above. Explain the trend by referring to the values of $P_c(x)$ and $n(x)$ for such x.

d. What is the theoretical domain of $R(x)$?

Domain and Asymptotes The unusual pattern of values for the profit-per-ticket function at and on either side of $x = 30$ can be explained by analyzing the domain of $R(x)$ and values of the numerator and denominator functions used to define $R(x)$. That is the sort of analysis required to understand any rational function and its graph.

5 Study this graph of the rational function $f(x) = \dfrac{x - 1}{x^2 - 2x - 3}$.



The graph has the lines $x = -1$ and $x = 3$ as **vertical asymptotes**. As values of x approach -1 from below, the graph suggests that values of $f(x)$ decrease without lower bound. As values of x approach -1 from above, the graph suggests that values of $f(x)$ increase without upper bound.

Describe the pattern of change in function values as x approaches 3.

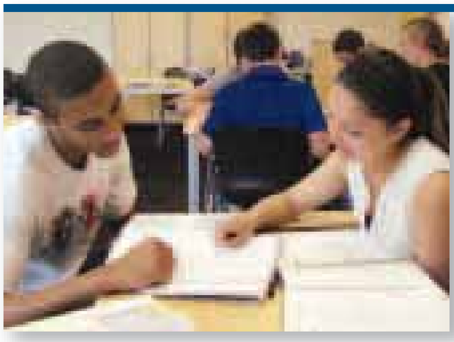
6 Inspect values of $f(x) = \dfrac{x - 1}{x^2 - 2x - 3}$ in terms of values of the numerator $(x - 1)$ and values of the denominator $(x^2 - 2x - 3)$ to help answer these questions about the function and its graph.

a. How could you have located the x-intercept of the graph by examining the rational expression that defines $f(x)$?

b. How could you have located the two vertical asymptotes of the graph by studying the rational expression that defines $f(x)$?

c. What is the domain of $f(x)$—for what values of x can a value of $f(x)$ be calculated and for what values is it impossible to calculate $f(x)$? Explain how your answer to that question is related to your answer to the question in Part b.

d. The graph of $f(x)$ has the x-axis as a **horizontal asymptote**. As values of x decrease without lower bound and increase without upper bound, the graph of $f(x)$ gets closer and closer to the x-axis but never reaches it. How could you have located that horizontal asymptote of the graph by studying the rational expression that defines $f(x)$?



7 For each of the following rational functions, use algebraic reasoning to:

- find coordinates of all x-intercepts and the y-intercept of its graph.
- find equations of all vertical and horizontal asymptotes.
- describe the domain of the function.
- sketch a graph on which you label intercepts and asymptotes.

Then check by examining a graph of the function, and correct your responses if necessary. Be prepared to explain how to avoid any errors you made.

a. $g(x) = \frac{3}{x - 2}$

b. $h(x) = \frac{2x + 3}{x^2 - 9}$

c. $j(x) = \frac{2x + 1}{3x}$

d. $k(x) = \frac{x^2 + 1}{x + 1}$

e. $m(x) = \frac{6x + 1}{3x + 5}$

f. $s(x) = \frac{2x^2 + 2x + 3}{x + 1}$

8 What strategies for analyzing features of a rational function and its graph are suggested by your work on Problems 1–7?

Concert Profit Prospects Reprise One of the important statistics used to describe the profitability of any business venture is the ratio of profit to income. For example, if a business earns profit of \$8 million on income from sales of \$100 million, it would report an 8% ratio of profit to income because 8 ÷ 100 = 8%.

Now recall the profit and income functions derived in analyzing the concert business. Profit from ticket sales for the concert (excluding snack bar operations) is a function of ticket price x with rule $P_c(x) = -25x^2 + 875x - 4{,}750$. Income from concert ticket sales is also a function of ticket price with rule $t(x) = -25x^2 + 750x$.

9 Consider the function $S(x)$ that gives the ratio of profit to income from concert ticket sales as a function of ticket price.

a. Write an algebraic rule for $S(x)$.

b. Will the graph of $S(x)$ have any vertical asymptotes? If so, where?

c. Sketch a graph of $S(x)$ for domain values from 0 to 30. Label the vertical asymptotes with their equations.

d. Describe the behavior of the profit-to-income function near each of the vertical asymptotes. Then use the rule for $S(x)$ to explain why that behavior occurs.

e. Estimate coordinates of any x-intercepts for $S(x)$. Explain what the x-intercepts tell about the concert profit-to-income ratio as a function of ticket price.

f. Estimate the ticket price that would maximize the profit to income ratio. Label that point on your graph of $S(x)$.

Summarize the Mathematics

In this investigation, you derived and analyzed rules for two rational functions used to model conditions in operation of a concert business. Then you studied a variety of other rational functions to generalize strategies for finding critical points and sketching graphs.

a. What can cause a rational function to be undefined at particular values of x?

b. Describe the behavior of a rational function and its graph near vertical and horizontal asymptotes.

c. How can you analyze the numerator and denominator in a rational expression to locate zeroes of the corresponding rational function? The intercepts and asymptotes of its graph?

Be prepared to explain your ideas to the class.

Check Your Understanding

Use your understanding of rational functions to answer the following questions about the relationship of concert profit $P_c(x) = -25x^2 + 875x - 4{,}750$ to concert operating expenses $c(x) = 4{,}750 - 125x$.

a. Write an algebraic rule for the function $T(x)$, which gives the ratio of profit to operating expenses, that is, the number of dollars earned for every dollar spent by the concert promoters.

b. Find the location of all vertical asymptotes for $T(x)$. Describe the behavior of the function near those asymptotes. Explain why that behavior occurs.

c. What is a reasonable domain for $T(x)$ when it is being used to model conditions in the concert business?

d. At about what ticket price is the ratio of profit to operating expenses maximized?

Investigation 2 Simplifying Rational Expressions

When work on mathematical problems leads to fractions like $\frac{8}{12}$ or $\frac{12}{18}$ or $\frac{9}{15}$, you know that it is often helpful to replace each fraction by a simpler equivalent fraction. For example, $\frac{8}{12} = \frac{4 \cdot 2}{4 \cdot 3} = \frac{4}{4} \cdot \frac{2}{3} = 1 \cdot \frac{2}{3}$, so $\frac{8}{12} = \frac{2}{3}$.

As you work on the problems of this investigation, look for answers to these questions:

What principles and strategies help to simplify rational expressions?

What cautions must be observed when simplifying rational expressions?

1. Consider again the ticket sale income function $t(x) = -25x^2 + 750x$ and the function $n(x) = -25x + 750$, giving number of tickets sold. In both functions, x represents the price per ticket in dollars.

   a. Write an algebraic rule for the rational function $U(x)$, giving income per ticket sold.

   b. Examine the values of $U(x)$ for integer values of x from 0 to 30. Explain why the pattern in those results makes sense when you think about what the expressions in the numerator and denominator represent in the problem situation.

   c. Factor both numerator and denominator of the expression defining $U(x)$. Then write the simpler rational expression that results from removing common factors.

   d. How does the result in Part c relate to the pattern you observed in Part b?

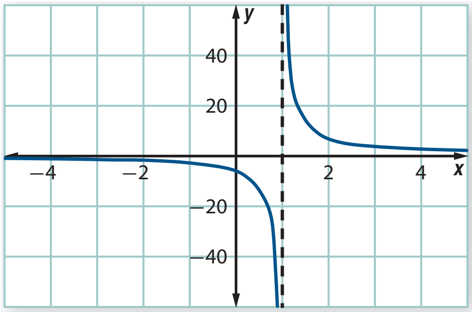
   e. When Rashid found the simplified expression called for in Part c, he said it is *equivalent* to the original rational expression. Flor said she did not think so, because the domain for $U(x)$ is different from that of the simplified expression. Who is correct?

2. Consider next the function $f(x) = \dfrac{x^2 + 2x - 15}{x^2 - 4x + 3}$.

   a. Simplify $\dfrac{x^2 + 2x - 15}{x^2 - 4x + 3}$ by first factoring both the numerator and denominator. Let $g(x)$ be the new function defined by that simplified expression.

   b. Study tables of values for $f(x)$ and $g(x)$ for integer values of x in the interval $[-10, 10]$. Find any values of x for which $f(x) \neq g(x)$. Explain why those inputs produce different outputs for the two functions.

   c. Use a calculator or computer to draw graphs of $f(x)$ and $g(x)$ over the interval $[-10, 10]$. Compare the graphs.



   d. Now look again at tables of values for the two functions with smaller step sizes to see if you can explain why the differences observed in Part b do not seem to show up in the graphs.

3. For each of the following rational functions, use factoring to produce simpler and nearly equivalent rules. In each case, note any values of x for which the original and the simplified rules will not produce the same output values. Check your results by examining tables of values and graphs of the functions.

   a. $f(x) = \dfrac{2x - 10}{x - 5}$

   b. $g(x) = \dfrac{x + 3}{2x^2 + 6x}$

   c. $h(x) = \dfrac{x^2 + 7x + 12}{x + 3}$

   d. $j(x) = \dfrac{x^2 + 7x + 12}{x^2 + 5x + 6}$

4 In Problem 3, you examined rational functions that had zeroes in the denominator but did not have the expected vertical asymptotes at some of those domain points. Study those examples and your simplifications again to formulate an answer to this question.

In what cases will a zero of the denominator in a rational function not indicate a vertical asymptote of its graph?

5 When you are faced with the task of simplifying rational expressions, you may be tempted to make some "cancellations" that do not produce nearly equivalent results. For example, when Arlen and Tara were faced with the function $s(t) = \frac{t + 9}{6t + 9}$, they produced what they thought were equivalent simpler expressions for the rule.

a. Arlen came up with $s_1(t) = \frac{1}{6}$. How could you show that this new function is quite different from the original? How could you help Arlen clarify his reasoning?

b. Tara came up with $s_2(t) = \frac{0}{6}$. How could you show that this new function is quite different from the original? How could you help Tara clarify her reasoning?

6 To check your skill and understanding in simplifying rational expressions, follow these steps on your own.

(1) Write an algebraic fraction in which both the numerator and the denominator are different linear expressions.

(2) Create another linear expression and multiply both numerator and denominator by that expression. Then expand the resulting numerator and denominator into standard form quadratic polynomials.

(3) Exchange the resulting rational expression with a classmate. See if you can find the quotient of linear expressions with which each began.

Summarize the Mathematics

In this investigation, you explored simplification of rational expressions.

a What is meant by simplifying rational expressions?

b For a given rational expression, what strategies can be applied to find another expression that is simpler but nearly equivalent to the original?

c What cautions must be observed when simplifying rational expressions?

Be prepared to explain your ideas to the class.

Check Your Understanding

Use your understanding of equivalent rational expressions to complete the following tasks.

a. The function $T(x) = \frac{-25x^2 + 875x - 4{,}750}{-125x + 4{,}750}$ compares concert profit to operating expenses.

i. Identify all values of x for which $T(x)$ is *undefined*.

ii. Explain why the expression defining $T(x)$ can be simplified to $\frac{x^2 - 35x + 190}{5x - 190}$.

iii. If the simplified expression in part ii were used to define a new function $K(x)$, would it have the same domain as $T(x)$?

b. Consider the expression $\frac{x^2 + 6x + 8}{x^2 - x - 6}$.

i. Identify all values of x for which this expression is undefined.

ii. Simplify the expression.

iii. Explain why simplifying this expression does or does not produce an expression with different undefined points than the original.

Investigation 3 Adding and Subtracting Rational Expressions

In Investigation 1, you derived the profit-per-ticket function $R(x) = \frac{-25x^2 + 875x - 4{,}750}{-25x + 750}$ for business at a concert venue. As you know, the concert promoters also make profit from sales at snack bars that operate before the concert and at intermissions. Including that profit source in the business calculations requires new operations on rational expressions and functions.

As you work on the problems of this investigation, look for answers to this question:

What principles and strategies guide addition and subtraction of rational expressions?

1 The concert promoter's business analysis suggests that total profit $P_b(x)$ from sale of snack bar items will be related to ticket price x by the function $P_b(x) = -175x + 5{,}250$.

a. Why does it make sense that profit from snack bar sales depends on ticket price?

b. Write and simplify a rule for the function $V(x)$ that gives the profit per ticket from snack bar sales. Remember that the number of people attending the concert is related to ticket price by the function $n(x) = -25x + 750$.

c. What are the values of $R(20)$, $V(20)$, and $R(20) + V(20)$? What do those values tell about the profit prospects for the whole concert event?

2 When the concert promoters asked their business manager to find a function $W(x)$ that shows combined profit per ticket from ticket and snack bar sales, she suggested two possibilities.

$$W_1(x) = \frac{P_c(x)}{n(x)} + \frac{P_b(x)}{n(x)} \quad \text{and} \quad W_2(x) = \frac{P_c(x) + P_b(x)}{n(x)}$$

a. Using the algebraic expressions for each function involved, you get:

$$W_1(x) = \frac{-25x^2 + 875x - 4{,}750}{-25x + 750} + \frac{-175x + 5{,}250}{-25x + 750}$$

$$\text{and} \quad W_2(x) = \frac{-25x^2 + 700x + 500}{-25x + 750}.$$

How could you convince someone that both ways of calculating total profit per ticket are correct by:

i. reasoning about the meaning of each expression in the concert business situation?

ii. using what you know about addition of arithmetic fractions like $\frac{2}{5} + \frac{4}{5}$?

b. Which of the two expressions for total profit per ticket would be better for:

i. showing how each profit source contributes to total profit per ticket?

ii. calculating total profit per ticket most efficiently?

c. Find what you believe to be the expression that is most efficient for calculating $W(x)$. Use that expression to find the profit per ticket from *both* ticket and snack bar sales when the ticket prices are \$10, \$15, \$20, and \$25.

d. Estimate the ticket price that will yield maximum profit per ticket for:

i. the concert operation alone, $R(x)$.

ii. the snack bar operation alone, $V(x)$.

iii. the combination of concert and snack bar operations, $W(x)$.

3 Consider the following two rational functions.

$$f(x) = \frac{3x^2 + 5x - 2}{4x + 1} \quad \text{and} \quad g(x) = \frac{x^2 + 4x + 4}{4x + 1}$$

a. Evaluate $f(x)$, $g(x)$, and $f(x) + g(x)$ for a variety of values of x.

b. If $h(x) = f(x) + g(x)$, find what you believe to be the simplest algebraic rule for $h(x)$.

c. Use the rule from Part b to calculate $h(x)$ for the same values of x you used in Part a. Then compare the results with what you obtained in Part a.

d. If you simplify the algebraic fraction that results from adding $f(x)$ and $g(x)$, there is one value of x for which care must be exercised in work with $h(x) = f(x) + g(x)$.

  i. What is that number? (*Hint:* Examine a table of values of $h(x)$ in increments of 0.25.)

  ii. Why do you get different results from using a simplified rule for $h(x)$ and the given rules for $f(x)$ and $g(x)$?

4 Using the rules for $f(x)$ and $g(x)$ given in Problem 3, let $k(x) = f(x) - g(x)$.

a. Find a rule in simplest form for $k(x)$.

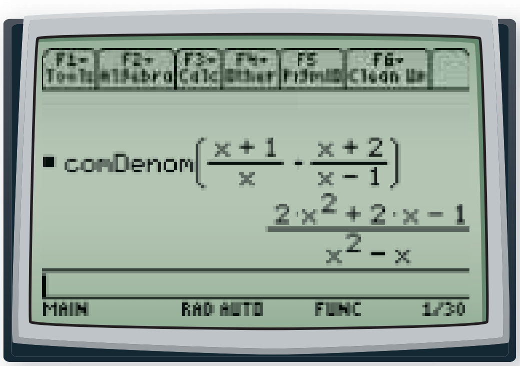
b. Check your answer to Part a by calculating $f(x) - g(x)$ and $k(x)$ for a variety of values of x.

c. How can you quickly tell that the numerator and denominator of $k(x)$ do not have a common factor?

Finding Common Denominators From your experiences with arithmetic, you know that addition and subtraction of fractions become more involved if the denominators are not the same. For example, to find the sum $\frac{2}{5} + \frac{7}{10}$, you first have to replace the addends with equivalent fractions having a common denominator. In this case, you might write $\frac{2}{5}$ as $\frac{4}{10}$ to get the sum $\frac{11}{10}$.

5 The same need for common denominators arises in work with algebraic fractions. And the strategies that work in dealing with numerical fractions can be applied to algebraic fractions as well.

For example, consider $\frac{x+1}{x} + \frac{x+2}{x-1}$. Examine the results produced by a CAS shown below.



a. Explain why the denominator of $x^2 - x$ is correct.

b. Why is the numerator $2x^2 + 2x - 1$?

6 Consider the following questions about airline flights from Chicago to Philadelphia and back. The cities are 678 miles apart. The return trip is scheduled to take almost half an hour longer than the flight from Chicago to Philadelphia because winds in the upper atmosphere almost always flow from west to east. Suppose the speed of the airplane without wind is s miles per hour, and the speed of the wind is 30 miles per hour from west to east.

a. Flight time is a function of distance and speed. Explain why the flight time from Chicago to Philadelphia is given by $f(s) = \frac{678}{s + 30}$ and the return time is given by $g(s) = \frac{678}{s - 30}$.

b. What information is given by values of $f(s) + g(s)$?

c. Explain why the rules for $f(s)$ and $g(s)$ can be replaced by

$$f(s) = \frac{678(s - 30)}{(s + 30)(s - 30)} \text{ and } g(s) = \frac{678(s + 30)}{(s - 30)(s + 30)}.$$

Why are those symbolic manipulations a useful idea?

d. Use the results in Part c to find a rule for $f(s) + g(s)$. Simplify it if possible.

e. Use the simplified rule for $f(s) + g(s)$ to find flight time for a round trip from Chicago to Philadelphia if the airplane's speed without wind is 310 miles per hour. Compare that result to the value of $f(310) + g(310)$.

CPMP-Tools

7 Adapt the reasoning used in work on Problems 5 and 6 and what you know about adding arithmetic fractions with unlike denominators to write each of the following algebraic expressions in equivalent form as a single fraction. Then simplify each result as much as possible. Check your reasoning with a CAS.

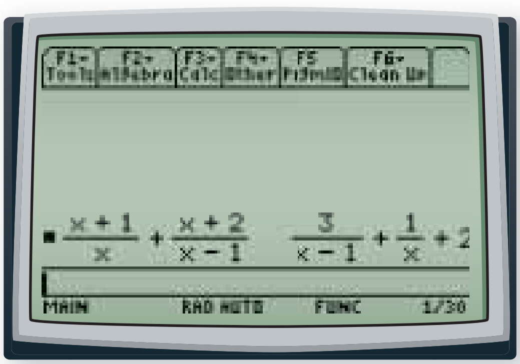
a. $\frac{3x + 1}{5} + \frac{x + 4}{x}$

b. $\frac{5}{x - 1} + \frac{x + 1}{x}$

c. $\frac{x + 2}{x + 3} + \frac{x + 1}{2x - 6}$

d. $3x + 1 + \frac{x + 4}{x}$

8 The following display shows how the sum of two fractions is calculated by a CAS *without* use of the **comDenom** command. Use algebraic reasoning to show that this result is equivalent to that obtained in Problem 5.



Summarize the Mathematics

In this investigation, you learned how to add and subtract rational expressions.

a Adding rational expressions is easy when the two expressions are related in a particular way. What is that relationship? If it holds, how do you add the expressions?

b If the most convenient relationship between rational expressions does *not* hold, what can you do to make addition or subtraction of the expressions possible?

Be prepared to explain your ideas to the class.

Check Your Understanding

Use your understanding of adding and subtracting rational expressions to find simplified rules for the sums and differences of functions in Parts a–c, where $f(x) = \frac{3x - 1}{4x}$, $g(x) = \frac{x - 5}{4x}$, and $h(x) = \frac{3x}{2x + 3}$.

a. $f(x) + g(x)$

b. $f(x) - g(x)$

c. $h(x) - g(x)$

Investigation 4 Multiplying and Dividing Rational Expressions

You have learned how to simplify, add, and subtract rational expressions, and you have seen how those operations are similar to operations with numerical fractions. The similarities continue, with some limitations, to multiplication and division.

As you work on the problems of this investigation, look for answers to these questions:

If the rules for two rational functions f(x) and g(x) are given, how can you calculate rules for the product and quotient of those functions?

What cautions must be observed when simplifying products and quotients of a rational expressions?

1 Give results, in simplest form, for these products of numerical fractions.

a. $\frac{5}{7} \bullet \frac{3}{10}$

b. $\frac{3}{5} \bullet \frac{9}{6}$

c. $\frac{5}{8} \bullet \frac{12}{25}$

2 Suppose that $f(x) = \frac{x^2}{2}$ and $g(x) = \frac{12}{x}$.

a. If $h(x) = f(x) \cdot g(x)$, what do you think will be the simplest expression that gives correct values of $h(x)$ for all x?

b. Test your idea in Part a by comparing tables of values for $\left(\frac{x^2}{2}\right)\left(\frac{12}{x}\right)$ and for your proposed simpler expression to calculate $h(x)$. Use values of x in the interval $[-10, 10]$.

c. Test your idea in Part a by graphing $y = \left(\frac{x^2}{2}\right)\left(\frac{12}{x}\right)$ for values of x in the interval $[-10, 10]$ and then graphing $y = h(x)$, using your proposed simplified expression for $h(x)$.

d. What similarities and what differences between multiplication of numeric and algebraic fractions are suggested by the results of your work on Parts a–c?

3 Suppose that $f(x) = \frac{3x}{x-2}$ and $g(x) = \frac{5x-10}{x}$.

a. If $h(x) = f(x) \cdot g(x)$, what do you think will be the simplest expression that gives correct values of $h(x)$ for all x?

b. Test your idea in Part a by comparing tables of values for $\left(\frac{3x}{x-2}\right)\left(\frac{5x-10}{x}\right)$ and for your proposed simpler expression to calculate $h(x)$. Use values of x in the interval $[-10, 10]$.

c. Test your idea in Part a by graphing $y = \frac{3x}{x-2} \cdot \frac{5x-10}{x}$ for values of x in the interval $[-10, 10]$ and graphing $y = h(x)$, using your proposed simplified expression for $h(x)$.

d. What similarities and what differences between multiplication of numeric and algebraic fractions are suggested by the results of your work on Parts a–c?

4 The following work shows how Cho calculated the product $h(x) = f(x) \cdot g(x)$ in Problem 3.

$$\begin{aligned}\frac{3x}{x-2} \cdot \frac{5x-10}{x} &= \frac{3x(5x-10)}{(x-2)x}\\ &= \frac{15x^2-30x}{x^2-2x}\\ &= \frac{15x(x-2)}{x(x-2)}\\ &= 15\end{aligned}$$

a. Explain how Cho could have saved time by factoring and removing identical factors before expanding the numerator and denominator expressions.

b. What restrictions on the domain of $f(x)$ and $g(x)$ has Cho ignored? What are the consequences?

5 Simplify $\frac{x-2}{x^2-25} \cdot \frac{2x^2+11x+5}{6x-12}$ by first factoring and then removing common factors before expanding the products in the numerator and denominator. Check your answer with a CAS. Be careful to explain the values of x for which the simpler expression is not equivalent to the original.

6 Suppose that $f(x) = \frac{6}{x}$ and $g(x) = \frac{2}{3x}$.

a. If $h(x) = f(x) \div g(x)$, what do think will be the simplest rule that gives correct values of $h(x)$?

b. Test your idea in Part a by comparing tables of values for $\frac{6}{x} \div \frac{2}{3x}$ and for your proposed simpler expression to calculate $h(x)$. Use values of x in the interval $[-10, 10]$.

c. Test your idea in Part a by graphing $y = \frac{6}{x} \div \frac{2}{3x}$ for values of x in the interval $[-10, 10]$ and then graphing $y = h(x)$, using your proposed simplified expression for $h(x)$.

d. What similarities and what differences between division of numeric and algebraic fractions are suggested by the results of your work on Parts a–c?

7 Suppose that $f(x) = \frac{2x}{x^2-9}$ and $g(x) = \frac{5x^2}{2x-6}$.

a. If $h(x) = f(x) \div g(x)$, what do you think will be the simplest rule that gives correct values of $h(x)$?

b. Test your idea in Part a by comparing tables of values for $\frac{2x}{x^2-9} \div \frac{5x^2}{2x-6}$ and for your proposed simpler expression to calculate $h(x)$. Use values of x in the interval $[-10, 10]$.

c. Test your idea in Part a by graphing $y = \frac{2x}{x^2-9} \div \frac{5x^2}{2x-6}$ for values of x in the interval $[-10, 10]$ and then graphing $y = h(x)$, using your proposed simplified expression for $h(x)$.

d. What similarities and what differences between division of numeric and algebraic fractions are suggested by the results of your work on Parts a–c?

8 Consider the rational expression $\frac{6x^2+9x+8}{3x}$.

a. Which of these expressions (if any) are equivalent to the original? How do you know?

- **i.** $2x + \frac{9x+8}{3x}$
- **ii.** $2x + 9x + 8$
- **iii.** $2x + 3 + \frac{8}{3x}$

b. Check your answer to Part a by comparing graphs of the functions defined by each expression in the window $-5 \le x \le 5$ and $-10 \le x \le 20$.

Summarize the Mathematics

In this investigation, you explored ways that algebraic reasoning can be used to work with multiplication and division of rational expressions and functions.

a If you are given two rational functions $f(x) = \frac{a(x)}{b(x)}$ and $g(x) = \frac{c(x)}{d(x)}$, how would you find the simplest form of the rules for these related functions?

i. $h(x) = f(x) \cdot g(x)$

ii. $j(x) = f(x) \div g(x)$

b How can rational functions with rules in the form $r(x) = \frac{ax^2 + bx + c}{x}$ be written in equivalent form as separate fractions?

Be prepared to explain your ideas to the class.

Check Your Understanding

Use your understanding of multiplication, division, and equivalent expressions to complete these tasks.

a. If $r(x) = \frac{x^2 + 3x - 10}{2x - 4}$ and $s(x) = \frac{3x - 6}{x - 2}$, find rules for the following expressed in simplest form.

i. $r(x) \cdot s(x)$

ii. $r(x) \div s(x)$

iii. $r(x) + s(x)$

iv. $r(x) - s(x)$

b. Write the rule for $y = \frac{3x - 5}{2x}$ in equivalent form as a sum of two separate fractions.

On Your Own

Applications

1 The Cannery designs and manufactures cans for packaging soups and vegetables. A&W Foods contracted with the Cannery to produce packaging for its store brand canned corn. Each can is to hold 500 cm^3 of corn. The company is interested in minimizing the cost of each can by minimizing the surface area of the can.

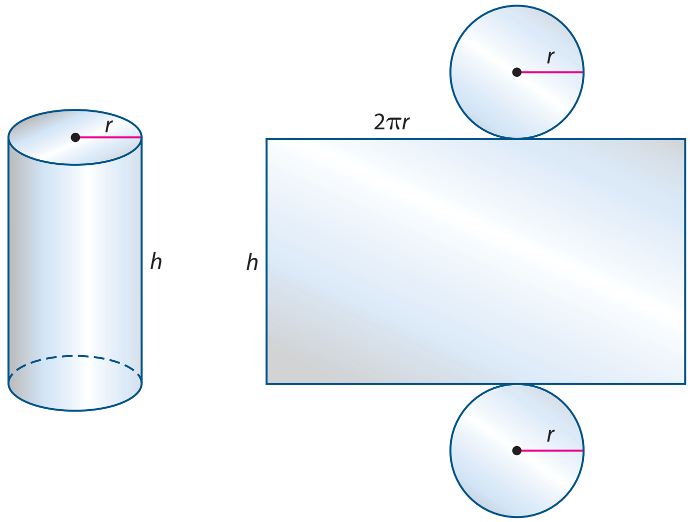
a. Using the information below about the volume of the can, find a rational function $h(r)$ that gives the height h of the can as a function of the radius r of the can.

Volume of a Can

$V = \pi r^2 h$

Surface Area of a Can

$SA = 2\pi r^2 + 2\pi rh$



b. Based on your results from Part a, write a function $S(r)$ to calculate the surface area of the can for any given radius r.

c. Determine a practical domain for $S(r)$. Sketch a graph of the function on that domain. Identify any vertical asymptotes.

d. Determine the minimum value of $S(r)$ on this practical domain. Give the radius and height of the can with minimum surface area to the nearest tenth of a centimeter.

2 At the concert venue you have studied several times in this unit, concert attendance, snack bar income, operating expenses, and profit are all functions of the ticket price x.

- Concert attendance is given by $n(x) = 750 - 25x$.
- Snack bar income is given by $s(x) = 7{,}500 - 250x$.
- Snack bar operating expenses are given by $b(x) = 2{,}250 - 75x$.

a. Write two equivalent rules for the function $P_b(x)$ that gives snack bar profit as a function of ticket price. Give one rule that shows the separate expressions for calculating snack bar income and operating cost and another that is equivalent, yet easier to use in calculations.

b. For the rational functions described in parts i–iv:

- explain what they tell about the business situation.
- describe the theoretical and practical domains of the functions—the values of x for which a function output *can be* calculated and the values of x for which calculation of outputs makes sense in the problem situation.
- sketch graphs of each function, indicating any vertical asymptotes and enough of the graph so that its overall shape is clear.

i. $\dfrac{s(x)}{n(x)}$ **ii.** $\dfrac{b(x)}{n(x)}$

iii. $\dfrac{P_b(x)}{s(x)}$ **iv.** $\dfrac{b(x)}{s(x)}$

3 Simplify the rules for the rational functions defined in Task 2 as far as possible. In each case, check to see whether the domain of each simplified function rule is the same as that of the original.

4 When income from concert ticket and snack bar sales (Task 2) are combined, the functions that model the business situation are:

- Total income as a function of average ticket price is given by $I(x) = -25x^2 + 500x + 7{,}500$.
- Combined operating expenses for the concert and the snack bar is given by $E(x) = 7{,}000 - 200x$.
- The number of tickets sold is given by $n(x) = 750 - 25x$.

a. Write two equivalent rules for the function $P(x)$ that gives total profit as a function of ticket price. Give one rule that shows the separate expressions for calculating income and cost and another that is equivalent, yet easier to use in calculations.

b. For the rational functions described in parts i–iv:

- explain what they tell about the business situation.
- describe the theoretical and practical domains of the functions—the values of x for which a function output *can be* calculated and the values of x for which calculation of outputs makes sense in the problem situation.
- sketch graphs of each function, indicating any vertical asymptotes and enough of the graph so that its overall shape is clear.

i. $\dfrac{P(x)}{I(x)}$ **ii.** $\dfrac{E(x)}{I(x)}$

iii. $\dfrac{P(x)}{n(x)}$ **iv.** $\dfrac{E(x)}{n(x)}$

5 Simplify the rules for the rational functions defined in Task 4 as far as possible. In each case, check to see whether the domain of the simplified function rule is the same as that of the original.

6 Describe the domains of these rational functions. Explain how you know that your answers are correct.

a. $f(x) = \dfrac{4x + 1}{3x + 2}$ **b.** $g(x) = \dfrac{5x + 10}{x + 2}$

c. $h(x) = \dfrac{x + 1}{x^2}$ **d.** $j(x) = \dfrac{x^2 + 3}{x^2 - 9}$

7 Simplify these algebraic fractions as much as possible. In each case, check to see whether the simplified expression is undefined for the same values of x as the original expression.

a. $\dfrac{4x + 12}{8x + 4}$ **b.** $\dfrac{4 - 3x}{6x - 8}$

c. $\dfrac{15x^2 + 6x}{3x^2}$ **d.** $\dfrac{4 + x^2}{16 - x^2}$

8 Write each of these sums and differences of rational expressions in equivalent form as a single algebraic fraction. Then simplify the result as much as possible.

a. $\dfrac{4x + 13}{x - 3} + \dfrac{x + 2}{x - 3}$ **b.** $\dfrac{3x + 7}{x - 2} - \dfrac{x + 5}{x - 2}$

c. $(x + 2) + \dfrac{x + 6}{x - 4}$ **d.** $\dfrac{3x^2 + 5x + 1}{x^2 - 25} - \dfrac{3x^2 + 4x - 4}{x^2 - 25}$

9 Write these products and quotients of rational expressions in equivalent form as single algebraic fractions. Then simplify the results as much as possible.

a. $\dfrac{2x}{x + 2} \cdot \dfrac{3x + 6}{x^2}$ **b.** $\dfrac{2x + 6}{x + 2} \cdot \dfrac{3x + 2}{x + 3}$

c. $\dfrac{x + 2}{x} \div \dfrac{3x + 6}{x^2}$ **d.** $\dfrac{2x}{x + 2} \div \dfrac{x^2}{2x + 4}$

10 Write each of these products and quotients of rational expressions in equivalent form as a single algebraic fraction. Then simplify the result as much as possible.

a. $\dfrac{2x + 4}{x^2 - 6x} \cdot \dfrac{x^2 - 36}{4x + 8}$ **b.** $\dfrac{x - 3}{7x} \cdot \dfrac{3x^2}{x^2 - 2x - 3}$

c. $\dfrac{x^2 - 4}{x^2 + 2x - 5} \div \dfrac{x + 2}{x^2 + 2x - 5}$ **d.** $\dfrac{2x^2 + 10x}{4x + 10} \div \dfrac{3x^2 + x}{9x + 3}$

11 Write each of these rational expressions as a sum or difference of two or more algebraic fractions in simplest form.

a. $\dfrac{3x + 5}{x}$

b. $\dfrac{3x^2 + 5x}{x}$

c. $\dfrac{3x^2 - 9x + 12}{6x^2}$

d. $\dfrac{3x + 21}{x + 7}$

Connections

12 Look back at your work for Applications Task 1.

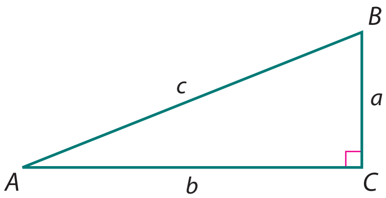
a. Record the height and radius of the can with minimum surface area in Part d.

b. Find the radius and height of the can with minimum surface area for cylindrical cans with these volumes: 236 cm^3, 650 cm^3, and 946 cm^3.

c. Plot (*optimal radius, optimal height*) for each of the cans in Parts a and b.

d. What relationship seems to exist between the optimal radius and height for a given volume?

13 In any right triangle ABC with right angle at C, the *tangent* of $\angle A$ is the ratio $\frac{a}{b}$.



Use what you know about operations with fractions to prove that for any acute angle A,

$$\frac{\sin A}{\cos A} = \tan A.$$

14 Consider the function $r(x) = \dfrac{3(2)^x}{0.5(10)^x}$.

a. Is $r(x)$ a rational function?

b. When the function is graphed, are there any asymptotes? If so, give their location.

c. Show how the expression defining $r(x)$ can be written in a simpler form that shows the function family to which it belongs.

15 The graphs of these functions all have *horizontal asymptotes*, lines with equations in the form $y = a$.

$f(x) = 2^x$ $\qquad g(x) = 0.5^x + 3$ $\qquad h(x) = \dfrac{1}{x}$

$j(x) = \dfrac{1}{x^2}$ $\qquad k(x) = \dfrac{8x + 5}{2x}$ $\qquad m(x) = \dfrac{x^2 + 2x - 1}{x^2 + x - 2}$

Study graphs and rules of the given functions to identify equations of their horizontal asymptotes.

16 Explain procedures that are necessary to perform these operations on arithmetic fractions.

a. Simplify a fraction like $\frac{36}{60}$.

b. Find the sum or difference of two fractions like $\frac{3}{5}$ and $\frac{7}{5}$.

c. Find the sum or difference of two fractions like $\frac{3}{5}$ and $\frac{7}{4}$.

d. Find the product of two fractions like $\frac{3}{5}$ and $\frac{7}{4}$.

e. Find the quotient of two fractions like $\frac{3}{5}$ and $\frac{7}{4}$.

17 The time required to complete a 100-mile bicycle race depends on the average speed of the rider. The function $T(s) = \frac{100}{s}$ tells time in hours if speed is in miles per hour.

a. Identify the vertical and horizontal asymptotes for the graph of $T(s)$.

b. Explain what each asymptote says about the way race time changes as average rider speed changes.

18 The intensity of sound from some source like a music speaker or a lightning strike is inversely proportional to the square of the distance from source to receiver. For example, suppose that the intensity of sound at various distances from a nearby explosion is given by the function $I(d) = \frac{100}{d^2}$, where distance is measured in meters and sound intensity in watts per square meter.

a. Identify the vertical and horizontal asymptotes for the graph of $I(d)$.

b. Explain what each asymptote says about the way received sound intensity changes as distance from the source changes.

19 When the Holiday Out hotel chain was planning to build a new property, organizers bought land for $5,000,000, and their architect made building plans that would cost $10,000,000 for the lobby, restaurant, and other common facilities. The architect also said that construction costs would average an additional $75,000 per guest room.

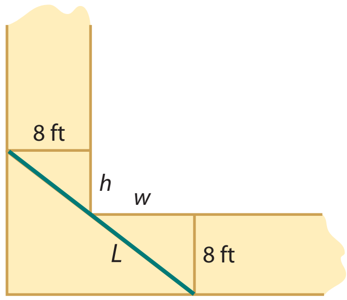
a. What rule defines the function that relates total construction cost C to the number n of guest rooms in the hotel?

b. What rule defines the function $A(n)$ that gives average cost per guest room as a function of the number n of guest rooms in the hotel?

c. What are the theoretical domains of the total cost and average cost functions? What are reasonable practical domains to consider for those functions?

d. Sketch a graph of the average cost per guest room function. Identify the vertical and horizontal asymptotes of that graph. Explain what the asymptotes tell about the way average cost per room changes as the number of planned guest rooms changes.

20 Rational functions arise naturally when working with similar triangles. Suppose that a long ladder is to be moved horizontally around a corner in a building with 8-foot hallways. The ladder can make it around the corner only if its length is less than all lengths L that satisfy the conditions of the diagram below.



a. Using similar triangles, determine a relationship between h and w.

b. Write a function relating the distance L to the lengths h and w. Then use the relationship from Part a to write L as a function of h. (Ignore the thickness of the ladder.)

c. Determine the minimum value $L(h)$.

d. As the longest possible ladder moves through the corner, what angle is made with the walls? Explain how this angle is determined by the symmetry of the situation.

e. What is the minimum value of L if one of the hallways is 4 feet wide and the other is 8 feet wide?

Reflections

21 Why does the term *rational function* seem appropriate for any function that can be expressed as the quotient of two polynomials?

22 Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function.

a. What can you conclude if $p(a) = 0$ and $q(a) \neq 0$?

b. What can you conclude if $q(a) = 0$ and $p(a) \neq 0$?

c. What can you conclude if $p(a) = 0$ and $q(a) = 0$?

23 What are some important things to remember and look for when:

a. analyzing the behavior of rational functions and their graphs?

b. simplifying rational expressions for functions?

c. adding or subtracting rational expressions and functions?

d. multiplying or dividing rational expressions and functions?

Extensions

24 Consider the function $f(x) = \dfrac{x^2 + x - 12}{x^2 - x - 6}$.

a. What are the values of x for which $f(x)$ is undefined?

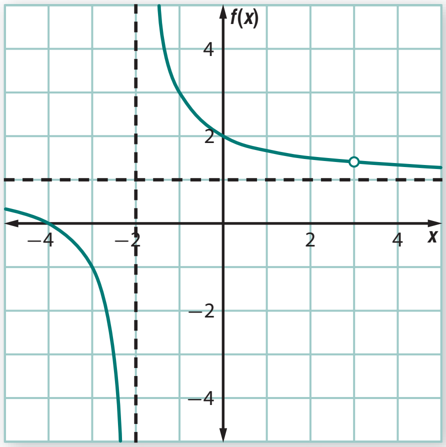
b. Graph $f(x)$ to see whether there seem to be vertical asymptotes at the values of x for which $f(x)$ is undefined.

c. Let $\tilde{f}(x)$ be the function resulting from simplifying the expression for $f(x)$. For how many values of x is $\tilde{f}(x)$ undefined?

d. How many vertical asymptotes does the graph of $\tilde{f}(x)$ have? Where are they?

e. How do the graphs of $f(x)$ and $\tilde{f}(x)$ compare?

25 The function $f(x)$ in Task 24 is *undefined* at $x = 3$. The nearly identical function $\tilde{f}(x)$ with rule obtained by "simplifying" the rational expression for $f(x)$, is *defined* at $x = 3$. However, the graph of $f(x)$ has no vertical asymptote at $x = 3$. That fact is shown by drawing a "hole" in the graph where $x = 3$.



The function $f(x)$ is said to have a **removable discontinuity** at $x = 3$ because the discontinuity or break in the graph can be removed by simplifying the expression. As a result, 3 is in the domain of $\tilde{f}(x)$ but not in the domain of $f(x)$. However, the other value of x for which $f(x)$ is undefined gives rise to a vertical asymptote, a discontinuity that cannot be removed by simplification. Therefore, $f(x)$ is said to have an **essential discontinuity** at $x = -2$.

Determine the essential and removable discontinuities for each function in Parts a–d.

a. $f(x) = \dfrac{4x + 12}{8x + 4}$

b. $g(x) = \dfrac{5x + 10}{x + 2}$

c. $h(x) = \dfrac{15x^2 + 6x}{3x^2}$

d. $j(x) = \dfrac{x^2 + 3}{x^2 - 9}$

26 Consider $f(x) = \dfrac{20x^3 + 10x^2 - 150x}{20x^2 - 125}$.

a. Simplify the expression for $f(x)$.

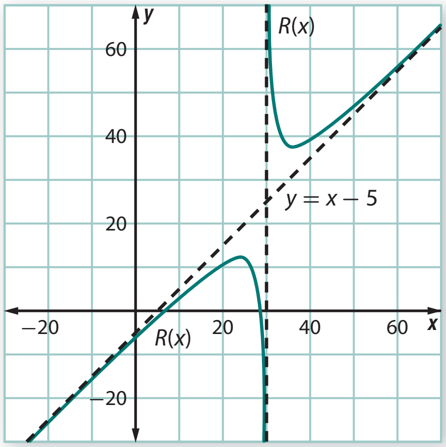
b. Compare graphs of $f(x)$ and the function given by the simplified expression.

c. Give the values of x for which $f(x)$ is undefined. Classify each discontinuity as removable or essential. (See Task 25.)

d. What is the domain of $f(x)$?

e. What is the domain of the function derived by simplifying the expression for $f(x)$?

27 The graph of $R(x) = \dfrac{-25x^2 + 875x - 4{,}750}{750 - 25x}$ has what is called an **oblique asymptote**. The next diagram shows how the graph of $R(x)$ approaches the oblique asymptote $y = x - 5$.



Use a spreadsheet or function table tool to examine values of $R(x)$ for very large values of x, say larger than 100, and then for "large negative" values of x.

a. Explain how your results illustrate the fact that as x approaches positive infinity, the graph of $R(x)$ gets arbitrarily close to the line $y = x - 5$, while remaining above it.

b. Explain how your results illustrate the fact that as x approaches negative infinity, the graph of $R(x)$ gets arbitrarily close to the line $y = x - 5$, while remaining below it.

28 Consider the function with rule $h(x) = 3x - 4 + \frac{5}{x}$.

a. What is the oblique asymptote for the graph of this function? How can you tell by analyzing values produced by the function rule when x is very large (approaches positive infinity) or very small (approaches negative infinity)?

b. How can the rule for $h(x)$ be given by an equivalent rational expression?

29 Suppose $f(x) = x - 3$ and $g(x) = \frac{6}{x - 2}$.

a. If you want to think of $f(x)$ as a rational function, what is its denominator?

b. What could you multiply that denominator by so it is the same as the denominator of $g(x)$?

c. Use your idea from Part b to find a rule for $f(x) - g(x)$ expressed with one simplified algebraic fraction.

d. Check your answer to Part c by graphing the simplified algebraic fraction and the function $y = (x - 3) - \frac{6}{x - 2}$.

e. The graph of $y = (x - 3) - \frac{6}{x - 2}$ has an oblique asymptote. Which form of the difference expression makes it easier to identify the equation of that asymptote, the expression $(x - 3) - \frac{6}{x - 2}$ or your answer to Part c? Be prepared to explain your reasoning.

30 Paco decided to spend the weekend at the local beach. He left on Friday, and his average speed was only 30 miles per hour. He took the same route home but was able to wait until Tuesday when the traffic was lighter. For the return trip, his average speed was 60 miles per hour.

a. What was Paco's average speed for the entire trip?

b. Consider the similar situation of Tory's trip to the beach. The entire trip took her two hours. For the first hour, her average speed was 60 miles per hour. During the second hour, she encountered traffic, so her average speed was 30 miles per hour for that hour of travel. What was Tory's average speed for the entire trip?

c. What information regarding Tory's trip is different from the information about Paco's trip? Which situation is in line with your definition of "average"?

Review

31 Solve these equations.

a. $6x - 14 = 16$

b. $12c + 18 = 24$

c. $6d + 5 = 4d + 7$

d. $7x + 4 = 9x + 13$

e. $5(2f - 3) + 6 = 10f - 9$

f. $5(2x + 5) = 10x + 15$

32 Rewrite each expression in a simpler equivalent form.

a. $\frac{108}{16}$

b. $\frac{3x + 12}{3}$

c. $\frac{10x^2 - 8}{24}$

33 Billy is standing 80 feet from the base of a building. The angle of elevation to the top of the building is 35°. If the distance from Billy's eyes to the ground is five feet, how tall is the building?

34 Consider the following statement.

The diagonals of a rhombus are perpendicular.

a. Write this statement in if-then form.

b. Is this a true statement? Explain your reasoning.

c. Write the converse of this statement.

d. Is the converse of this statement true? Explain your reasoning.

35 Solve these equations for x.

a. $2^x = 8$

b. $2^x = \frac{1}{16}$

c. $10^x = 10{,}000$

d. $10^x = 0.0000001$

36 Express each of these sums or differences as a single fraction in simplest form.

a. $\frac{1}{8} + \frac{5}{8}$

b. $\frac{4}{5} - \frac{2}{3}$

c. $\frac{1}{3} + \frac{5}{6}$

d. $\frac{3}{10} - \frac{7}{6}$

e. $\frac{2}{x} + \frac{3}{4}$

37 Sketch the solution set for each system of inequalities.

a. $\begin{cases} 3x + 6y \geq 12 \\ y \leq -5x + 10 \end{cases}$

b. $\begin{cases} y > x^2 \\ y < x + 2 \end{cases}$

38 Suppose that Elizabeth rolls two regular dice and finds the sum of the numbers that are showing.

a. What is the probability that the sum is even or greater than 8?

b. What is the probability that the sum is even and greater than 8?

c. If she rolls the dice twice what is the probability that the sum is greater than 8 on both rolls?

39 Express each of these fraction products and quotients as a single fraction in simplest form.

a. $\frac{1}{2} \cdot \frac{1}{3}$

b. $\frac{2}{3} \div \frac{4}{5}$

c. $\frac{3}{4} \cdot \frac{2}{3}$

d. $\frac{2}{3} \div \frac{1}{2}$

e. $\frac{a}{b} \cdot \frac{c}{d}$

40 In $\triangle ABC$ and $\triangle DEF$, $\angle B$ and $\angle E$ are right angles. In each case below, decide whether the given information is sufficient to conclude that $\triangle ABC \cong \triangle DEF$. If so, explain why. If not, give a counterexample.

a. $\overrightarrow{AC} \cong \overrightarrow{DF}$; $\angle C \cong \angle F$

b. $\angle A \cong \angle D$; $\angle C \cong \angle F$

c. $\overrightarrow{AC} \cong \overrightarrow{DF}$; $\overrightarrow{AB} \cong \overrightarrow{DE}$

41 Triangle ABC is an isosceles triangle with $AB = BC$.

a. If $m\angle B = 138°$, find $m\angle C$ and $m\angle A$.

b. If $AC = 16$ cm, find the perimeter and the area of $\triangle ABC$.

LESSON 4

Looking Back

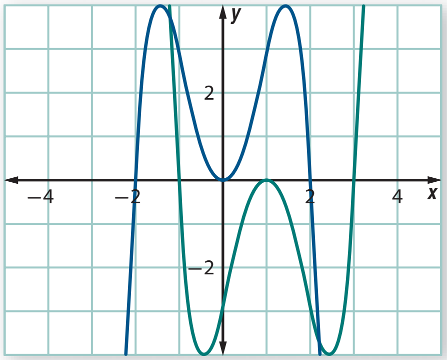
The lessons of this unit extended your knowledge and skill in work with algebraic functions and expressions in three ways. First, you learned how to use polynomials that go beyond linear and quadratic functions and expressions to model data and graph patterns that are more complex than the familiar straight lines and parabolas. You discovered ways of predicting the shape of polynomial function graphs from analysis of their rules—especially the number of local maximum and local minimum points and the number of zeroes—and ways of combining polynomials by addition, subtraction, and multiplication.

Second, you revisited quadratic functions and expressions to learn about completing the square to produce vertex forms for their rules. You learned how to use the vertex form to locate maximum and minimum points on the graphs of quadratic functions and how to solve equations by algebraic reasoning. You also studied the way that completing the square can be used to prove the quadratic formula for solving equations.

Finally, you explored properties and applications of rational functions and expressions. You learned how to identify undefined points of algebraic fractions and asymptotes of their graphs. You explored simplification of algebraic fractions and the ways that cautious use of such algebraic work can provide insights into functions and the problems they model. Then you learned how to combine rational functions and expressions by addition, subtraction, multiplication, and division.

The following tasks give you an opportunity to review and to put your new knowledge of polynomial and rational functions to work in some new contexts.

1 Cassie and Cliff decided to start a company called Maryland Web Design that would help individuals build personal sites on the World Wide Web. Cassie sketched out a possible logo for their company on graph paper. Cliff thought it looked like a graph of two polynomial functions, so he drew in a pair of axes.



a. What is the smallest possible degree of each polynomial function?

b. Write possible functions to describe the curves Cassie drew. Use $M(x)$ to describe the curve that looks like the letter M and $W(x)$ to describe the curve that looks like the letter W.

2 Cassie and Cliff decided that their company would specialize in designing Web logs, or "blogs." Since production of any blog requires about the same amount of programming time, each customer could be charged the same price.

Cassie and Cliff also estimated that the number of customers in a year would be related to the price by $n(x) = 600 - 2x$, where x is the price charged per blog.

Production of each blog will require about \$200 in programmer labor. To get started in business, they bought a computer for \$2,000.

a. Write rules for the following functions, simplifying where possible.

i. Annual income from blog designs $I(x)$ if the price is x dollars per blog

ii. Annual production costs for blog designs $C(x)$ if the price is x dollars per blog

iii. Annual profit from the business $P(x)$ if the price is x dollars per blog

iv. Profit per customer $P_c(x)$ if the price is x dollars per blog

b. What would be reasonable domains for the functions in Part a?

c. For what blog prices would the following functions be maximized?

i. Income $I(x)$

ii. Profit $P(x)$

iii. Profit per customer $P_c(x)$

d. What price would you recommend that Cassie and Cliff charge per blog? Explain your reasoning.

3 Find results of these algebraic operations. Express each answer in the equivalent form that would be simplest for evaluation at any specific value of x.

a. $(3x^4 + 5x^3 - 7x^2 + 9x - 1) + (2x^4 + 4x^2 + 6x + 8)$

b. $(3x^5 + 5x^3 - 7x^2 + 9x - 1) - (2x^4 + 4x^3 + 6x + 8)$

c. $(7x^2 + 9x - 1)(2x + 3)$

4 Look back at the graphs that look like an M and a W in Task 1.

a. Write and check the fit of models for the M whose function rules have these properties.

i. A product of linear factors that has zeroes at -2, 0, and 2

ii. A product of linear factors with zeroes at -2 and 2 and a repeated zero at $x = 0$

b. Write and check the fit of models for the W whose function rules have these properties.

i. A product of linear factors that has zeroes at -1, 1, and 3

ii. A product of linear factors with zeroes at -1 and 3 and a repeated zero at $x = 1$

c. Expand the products in Parts a and b that appear to give the best matches for the M and W drawings. Compare the resulting polynomials to the results produced in your work on Task 1. Reconcile any differences.

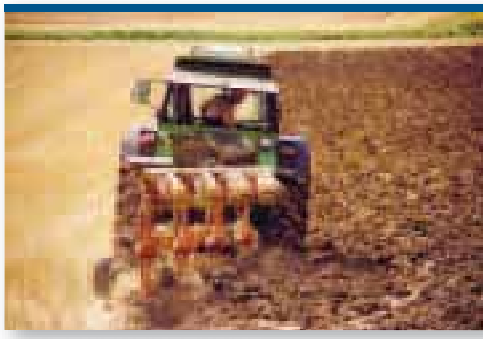
5 Express the quadratic function $f(x) = x^2 - 12x + 11$ in two ways, as the product of two linear factors and in vertex form. Then show how both forms can be used to find coordinates of these points on the graph of $y = f(x)$.

a. x-intercept(s)

b. y-intercept

c. maximum or minimum point

6 A farmer who grows wheat every summer has two kinds of expenses—*fixed costs* for equipment and taxes and *variable costs* for seed, fertilizer, and tractor fuel. Suppose that these costs combine to give total operating cost in dollars $C(x) = 25{,}000 + 75x$ for growing x acres of wheat.



a. What is a rule for the function $c(x)$ that gives the operating cost per acre on that farm?

b. The function giving operating cost per acre has a horizontal asymptote. Explore values of $c(x)$ as x increases to discover the equation of that asymptote.

c. Sketch a graph of $c(x)$. Explain what it shows about the way that operating cost per acre changes as the number of acres planted increases. In particular, explain what the horizontal asymptote tells about the relationship between number of acres planted and operating cost per acre.

7 Suppose that the wheat farmer in Task 6 can expect income of about \$125 per acre.

a. What functions give total profit $P(x)$ and profit per acre $p(x)$ from planting x acres of wheat?

b. How many acres of wheat must the farmer plant to break even on the crop?

c. Sketch a graph of the profit per acre function $p(x)$ for $0 < x < 2{,}000$. What seems to be the horizontal asymptote for this graph? What does it tell about the relationship between profit per acre and number of acres planted?

8 Consider the functions $f(x) = \dfrac{2x + 7}{x^2 - 4}$ and $g(x) = \dfrac{x + 5}{x^2 - 4}$.

a. Sketch graphs of each function. Identify any asymptotes.

b. Describe the domains of the two functions.

c. Calculate expressions in simplest form for these combinations of $f(x)$ and $g(x)$.

i. $f(x) + g(x)$

ii. $f(x) - g(x)$

iii. $f(x)g(x)$

iv. $f(x) \div g(x)$

Summarize the Mathematics

In this unit, you modeled problem situations using polynomial and rational functions and developed an understanding of some of the properties of those families of functions. You also learned how to add, subtract, and multiply polynomials and rational expressions.

a What kinds of functions are called:

- **i.** polynomial functions?
- **ii.** rational functions?

b Describe how you can determine:

- **i.** the degree of a polynomial.
- **ii.** local maximum or local minimum points on the graph of a polynomial function.
- **iii.** the zeroes of a polynomial function.
- **iv.** the sum or difference of two polynomials.
- **v.** the product of two polynomials.

c What is the vertex form of a quadratic function?

- **i.** What are the advantages and disadvantages of using this form?
- **ii.** How would you transform any quadratic expression in the form $ax^2 + bx + c$ into vertex form?

d For questions that call for solving quadratic equations $ax^2 + bx + c = 0$, how can you tell the number and kind of solutions from the value of the discriminant $b^2 - 4ac$?

e When working with rational expressions and functions, describe:

- **i.** when an expression will be undefined.
- **ii.** when the graph of a function will have a vertical asymptote.
- **iii.** when the graph of a function will have a horizontal asymptote.
- **iv.** strategies you can use to simplify the expression for a function.
- **v.** when two rational expressions can be added or subtracted and how you would go about performing the addition or subtraction.
- **vi.** how multiplication or division of two rational expressions can be performed.

Be prepared to share your examples and descriptions with the class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.